TITLE

BROWNIAN MOTION AND THE HEAT KERNEL ON
RIEMANNIAN MANIFOLDS

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BY

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CONTENTS

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DECLARATION

The material in this Thesis is original except where pointed out otherwise. The first Chapter gave rise to two publications: The first one was joint work with K.D. Elworthy and A. Truman which appeared in "From Local Times to Global Geometry, Control and Physics, ed. K.D. Elworthy p. 84-99, Pitman Research Notes in Mathematics, Series No. 150, Longman Scientific and Technical (1986). The second was independent work which appeared in the same Journal (p. 320-328). A third article, still from Chapter I, is submitted for publication in the "Proceedings of the Edinburgh Math. Society".

No part of this Thesis has been submitted for any degree at any other University or Centre of Learning.

Signed: .... Date: ....

GENERAL INTRODUCTION

The starting point of this work is [17] where K.D. Elworthy and A. Truman obtained a probabilistic representation of the Heat Kernel of a complete Riemannian manifold with a pole relative to the differential operator $\frac{1}{2}\Delta + V$ where Δ is the Laplace-Beltrami operator on M and V is a potential term on M supposed continuous and bounded above. The purpose of Chapter I is to generalize the above situation by obtaining a probabilistic representation of the Dirichlet Heat Kernel of any open set $U \subset M \setminus Cut(y)$ star-shaped from $y \in M$ and with smooth boundary where $Cut(y)$ is the cut-locus of M at $y \in M$. The Heat Kernel is relative to the more general differential operator $L = \frac{1}{2}\Delta + b + V$ where b is a smooth vector field on M. The techniques used to achieve this include those used in [17] and [37]. This also generalizes the Heat Kernel Formula obtained in [37]. As an important application, we are able to obtain Heat Kernel formulae for the Euclidean n-sphere $\mathbb{S}^n$ and the Real Projective Space $\mathbb{RP}^n$.

In Chapter II we generalize Geodesic Normal Coordinates to Fermi Coordinates defined by A. Gray in [20]. The techniques of Chapter I carry over to here where the notion of the Semi-Classical Brownian Riemannian Bridge from a point $x \in M \setminus Cut(y)$ to another point $y \in M$ is replaced by the more general notion of the Semi-Classical Brownian Riemannian Bridge from a point $x \in M_0$ to a submanifold $N \subset M_0$ where $(\exp_\nu^{-1}, M_0)$ is a Fermi Chart about the submanifold N. We obtain an Integral Formula for the Heat Kernel i.e. the probabilistic representation of the Integral of the Heat Kernel (supported by a function of compact support) over the submanifold N of M. The representation involves the submanifold Bridge Process mentioned above. Then applying the Integral Formula via a Riemannian submersion Theorem in [14] we obtain formulae of Heat Kernels for the 2n-dimensional Complex Projective Space $\mathbb{CP}^n$ and the 4n-dimensional Quaternionic Projective Space $\mathbb{HP}^n$. These formulae are given in terms of Euclidean Brownian Bridges.

Chapter III consists of an Exact and Asymptotic Expansion formulae for the Integral of the Heat Kernel (supported by a function of compact support) over the submanifold N. These provide a double generalization of the expansions in [37]: The pole y is generalized to a submanifold N of M and the manifold M itself is replaced by the tubular neighbourhood M_0 of N in M. The Chapter closes by combining the Integral Formula of Chapter II, the Expansion Formulae and a submersion Theorem (Theorem (3.2) of Chapter III) to obtain expansions for the density $\rho_t(-,-)$ of $\pi(z_t)$ when $\pi:M \to B$ is a submersion between manifolds and $(z_t)0 \leq t \leq +\infty$ is a diffusion process on M.

In Chapter IV we specialize the situation of Chapter III by taking $N = \{y\}$. Thus the tubular neighbourhood M_0 of N is reduced to the ordinary normal neighbourhood $U \subset M \setminus Cut(y)$. We thus recover the Heat Kernel Formula of Chapter I then an Exact and an Asymptotic Expansion Formulae (for small time) for the Dirichlet Heat Kernel $p_t^U(x,y)$ and for $p_t^M(x,y)$ (when x and y are not too far apart in M). Again this is a generalization of the situation in [37]. Then some applications of the Asymptotic Expansion are obtained: First we obtain identities by comapring coefficients of the small time Asymptotic Expansion of the Integral of the Heat Kernel and coefficients obtained by integrating term by term the small time Asymptotic Expansion of the Heat Kernel of a Cartan-Hadamard Manifold. Next we obtain some identities by comparing coefficients obtained by the above small time Asymptotic Expansion and Milson's formula for the Hyperbolic n-space. Lastly we have the H.P. McKean and I.M. Singer expansion ([30]) where we compute the first three coefficients of the expansion in terms of the curvature of the manifold at $y \in M$. A "raw" expression for the fourth coefficient is obtained to close the Chapter (and the Thesis).

CHAPTER I : THE HEAT KERNEL FORMULA IN A GEODESIC CHART

§0. INTRODUCTION

Let M be a connected complete n-dimensional Riemannian manifold. Let $L = \frac{\mu^2}{2}\Delta + \mu^2 b + \mu^2 V$ be a second order differential operator on M where Δ is the Laplace-Beltrami operator, b a smooth vector field and V a continuous potential term which we suppose bounded above. Lastly ∇ will denote the gradient operator on M.

Let $(\exp_y^{-1}, U)$ be a geodesic chart on M centred at $y \in M$ where U has smooth boundary and lies in a set star-shaped from y. Star-shaped here means that for each x in the set, there exists a unique geodesic joining x and y and lying entirely in the set.

Let $p_t^{U,\mu}(-,-)$ be the Dirichlet Heat Kernel of U relative to the operator L.

We will make the following notations:

$$\theta_y(x) = |\det_U T_v \exp_y|$$

where $\exp_y(v) = x$ i.e. θ_y is the Jacobian determinant of the exponential map $\exp_y$.

$$B_y(x) = \exp\left\{\int_0^1 \langle b(\gamma(s)), \dot{\gamma}(s)\rangle_{\gamma(s)} ds\right\} \tag{0.2}$$

where γ is the unique goedesic from $x \in U$ to y parametrized to take time 1

$$C_y(x) = B_y(x)\,\theta_y^{-\frac{1}{2}}(x) \tag{0.3}$$

$$q_t(x,y) = (2\pi t\mu^2)^{-n/2}\, C_y(x) \exp\{-\frac{d(x,y)^2}{2\mu^2 t}\} \tag{0.4}$$

where d is the distance compatible with the Riemannian metric on M.

Let Cut(y) be the Cut-locus of M at $y \in M$ and let $(x_s^{u,t})_{0 \le s < (t \wedge \zeta^u)}$ be the inhomogeneous process with generator $\mathcal{L} + A_s^u$ where A_s^u is defined in geodesic normal coordinates by:

$$A_s^\mu(x) = -\frac{x}{t-s} + \mu^2 \,\nabla \log C_y(x). \tag{0.5}$$

The process is started from $x \in M \setminus Cut(y)$ and ζ^μ is its first exit time from $M \setminus Cut(y)$.

We will be working on the usual probability space $(\Omega, \underline{F}, \underline{\underline{F}}_t, P_x)$, where $x \in M \setminus Cut(y)$ and $t \ge 0$.

Our purpose here is to obtain a formula for the Dirichlet Heat Kernel $p_t^{U,\mu}(x,y)$. We will then use the formula to compute the Heat Kernel for the standard n-sphere $S^n = (S^n(1), g_0)$. In particular we will obtain the Heat Kernel for the standard 3-sphere S^3. We will then deduce an identity linking the spherical harmonics (see e.g. [13], Chapter V) and certain homogeneous polynomials harmonic on $\mathbb{R}^4$ (see e.g. [7], Chapter III). In particular we will deduce an expression for $P_x(\zeta > t)$ where ζ is the first (random) time that the Bridge Process in S^3 hits the point anti-podal to y. This ties up well with a formula in [26].

We will lastly deduce a formula for the Heat Kernel of the Real Projective Space $\mathbb{R}P^n$ from that of S^n.

§1. THE HEAT KERNEL FORMULA FOR U

We will first assume that U has compact closure $\bar{U} \subset M \setminus Cut(y)$.

Let f^{λ} be the solution of the diffusion equation in U with Dirichlet boundary conditions:

$$\frac{\partial f_t^{\lambda}}{\partial t} = L f_t^{\lambda}$$

$$f_0^{\lambda} = (2\pi\lambda\mu^2)^{-n/2}\, T_o \exp\{-\frac{d(-,y)^2}{2\lambda\mu^2}\} \qquad (1.1)$$

$$f_t^{\lambda}(x) = 0 \;\forall\, x \in \partial U;\; t > 0$$

where T_o is of compact support in U with $T_o(y) = 1$ and ∂U is the boundary of U.

(1.1) Theorem

$$p_t^{U,\mu}(x,y) = \lim_{\lambda \downarrow 0} f_t^{\lambda}(x).$$

Proof.

$$f_t^{\lambda}(x) = \int_U f_0^{\lambda}(z) p_t^{U,\mu}(x,z)dz \qquad (1.2)$$

where dz denotes the volume element measure on M. Replacing f_0^{λ} by its value given in (1.1), we have:

$$f_t^\lambda(x) = (2\pi\lambda\mu^2)^{-n/2} \int_U T_o(z)\exp\{-\frac{d(z,y)^2}{2\lambda\mu^2}\} \; p_t^{U,\mu}(x,z)dz \tag{1.3}$$

$$= (2\pi\lambda\mu^2)^{-n/2} \int_{T_yM} T_o(\exp_y v)\exp\{-\frac{\|v\|^2}{2\lambda\mu^2}\} \; p_t^{U,\mu}(x,\exp_y v)\theta_y(v)dv \tag{1.4}$$

since T_o has compact support in U.

Setting $v = \mu\sqrt{\lambda}\omega$, (1.4) becomes:

$$(2\pi)^{-n/2} \int_{T_yM} T_o(\exp_y(\mu\sqrt{\lambda}\omega))\exp\{-\frac{\|\omega\|^2}{2}\}p_t^{U,\mu}(x,\exp_y(\mu\sqrt{\lambda}\omega))\theta_y(\mu\sqrt{\lambda}\omega)dw \tag{1.5}$$

$$\underset{\lambda\downarrow 0}{\to} \; p_t^{U,\mu}(x,y) \; T_o(y) = p_t^{U,\mu}(x,y).$$

A probablistic representation of the solution f_t^λ of the diffusion equation is given by:

$$f_t^\lambda(x) = E_x(\chi_{\tau^\mu > t} f_o^\lambda(x_t^\mu)\exp\{\int_0^t \mu^2 V(x_s^\mu)ds\}) \tag{1.6}$$

where τ^μ is the first exit time from U of the diffusion process (x_s^μ) $s \geq 0$ with generator $L_o^\mu = \frac{\mu^2}{2}\Delta + \mu^2 b$. By the Girsanov-Cameron-Martin-Feyman-Kac formula,

$$f_t^\lambda(x) = E_x(\chi_{\tau^{\lambda,\mu} > t} \; f_o^\lambda(x_t^{\lambda,\mu})M_t^\lambda \exp\{\int_0^t \mu^2 V(x_s^{\lambda,\mu})ds\}) \tag{1.7}$$

where

(i) For each $t > 0$, $(x_s^{\lambda,\mu})$ $0 \le s \le (t \wedge \zeta^{\lambda,\mu})$ is now a diffusion process with generator $L^\mu + A_s^{\lambda,\mu}$ where $A_s^{\lambda,\mu}$ is defined in geodesic normal coordinates by:

$$A_s^{\lambda,\mu}(x) = -\frac{x}{\lambda+t-s} + \mu^2 \nabla \log C_y(x). \tag{1.8}$$

Following [38], we call $x_s^{\lambda,\mu}$ the Semi-Classical Brownian Riemannian Bridge in $M \setminus Cut(y)$.

(ii) $\zeta^{\lambda,\mu}$ is the hitting time of the Cut-locus by the bridge process $x_s^{\lambda,\mu}$ started at $x \in U$ and $\tau^{\lambda,\mu}$ is its first exit time from U.

(iii) M_t^λ $0 \le t < +\infty$ is the exponential local martingale given by the S.D.E.

$$dM_s^\lambda = - M_s^\lambda \langle A_s^{\lambda,\mu}(x_s^{\lambda,\mu}), u_s^{\lambda,\mu}\, dB_s \rangle_{x^{\lambda,\mu}}$$

$$M_0^\lambda = 1 \tag{1.9}$$

$$\text{i.e. } M_t^\lambda = \exp\{-\frac{1}{\mu}\int_0^t \langle A_s^{\lambda,\mu}(x_s^{\lambda,\mu}), u_s^{\lambda,\mu}\, dB_s\rangle_{x^{\lambda,\mu}} - \frac{1}{2\mu^2}\int_0^t \| A_s^{\lambda,\mu}(x_s^{\lambda,\mu})\|^2 ds\} \tag{1.10}$$

where $(u_s^{\lambda,\mu})$ is the horizontal lift of $(x_s^{\lambda,\mu})$ on the orthonormal frame bundle O(M) of M.

(1.2) <u>Lemma</u>

$$M_t^\lambda = \left(\frac{\lambda}{\lambda+t}\right)^{n/2} \exp\{-\frac{d(x,y)^2}{2\mu^2(\lambda+t)} + \frac{d(x_t^{\lambda,\mu},y)^2}{2\lambda\mu^2}\}$$

$$\frac{C_y(x)}{C_y(x_t^{\lambda,\mu})} \exp\{\mu^2 \int_0^t \frac{(L-V)C_y(x_s^{\lambda,\mu})}{C_y(x_s^{\lambda,\mu})} ds\} .$$

Proof.

Define $Y_s^\lambda : M \to R$ by

$$Y_s^\lambda(x) = -\frac{d(x,y)^2}{2(\lambda+t-s)} + \mu^2 \log C_y(x),$$

then $\nabla Y_s^\lambda = A_s^{\lambda,\mu}$ and so by Itô's formula,

$$Y_t^\lambda(x_t^{\lambda,\mu}) = Y_o^\lambda(x) + \int_0^t \frac{\partial Y^\lambda}{\partial s}(x_s^{\lambda,\mu})ds + \mu\int_0^t \langle A_s^{\lambda,\mu}(x_s^{\lambda,\mu}),\ u_s^{\lambda,\mu}\, dB_s\rangle_{x^{\lambda,\mu}} \tag{1.11}$$

$$+ \int_0^t \langle \nabla Y_s^\lambda(x_s^{\lambda,\mu}), \mu^2 b(x_s^{\lambda,\mu}) + A_s^{\lambda,\mu}(x_s^{\lambda,\mu})\rangle_{x^{\lambda,\mu}} ds + \frac{\mu^2}{2}\int_0^t \Delta Y_s^\lambda(x_s^{\lambda,\mu})ds$$

a.s. Now, substituting for $-\frac{1}{\mu}\int_0^t \langle A_s^{\lambda,\mu}(x_s^{\lambda,\mu}), u_s^{\lambda,\mu} dB_s\rangle$

in (1.10) we have:

$$M_t^\lambda = \exp\{\frac{Y_o^\lambda(x)}{\mu^2} - \frac{Y_t^\lambda(x_t^{\lambda,\mu})}{\mu^2} + \frac{1}{\mu^2}\int_0^t \frac{\partial Y_s^\lambda}{\partial s}(x_s^{\lambda,\mu})ds$$

$$+ \frac{1}{2\mu^2}\int_0^t \|A_s^{\lambda,\mu}(x_s^{\lambda,\mu})\|^2 ds + \int_0^t \langle A_s^{\lambda,\mu}(x_s^{\lambda,\mu}),\ b(x_s^{\lambda,\mu})\rangle ds + \frac{1}{2}\int_0^t \Delta Y_s(x_s^{\lambda,\mu})ds \quad \text{a.s.} \tag{1.12}$$

Set $r(x) = d(x,y)$.

Now, $A_s^{\lambda,\mu}(x) = \nabla Y_s^\lambda(x) = -\frac{r(x)\nabla r(x)}{\lambda+t-s} + \mu^2\, \nabla \log C_y(x)$.

Hence, $\frac{1}{2\mu^2}\|A_s^{\lambda,\mu}(x)\|^2$

$$= \frac{r^2(x)}{2\mu^2(\lambda+t-s)^2} - \frac{r(x)}{\lambda+t-s}\langle \nabla r(x),\ \nabla \log C_y(x)\rangle + \frac{\mu^2}{2}\|\nabla \log C_y(x)\|^2$$

$$= \frac{r^2(x)}{2\mu^2(\lambda+t-s)^2} - \frac{r(x)}{\lambda+t-s}\frac{\partial}{\partial r}\log C_y(x) + \frac{\mu^2}{2}\|\nabla \log C_y(x)\|^2 . \tag{1.13}$$

Recall that if a C^2-function $f:M \to R$ depends only on $r(x)$, then

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \left(\frac{n-1}{r} + \frac{\partial}{\partial r}\log \theta_y\right)\frac{\partial f}{\partial r}$$

and hence,

$$\frac{1}{2}\Delta \gamma_s^\lambda(x) = -\frac{1}{4(\lambda+t-s)}\left[2 + \left(\frac{n-1}{r(x)} + \frac{\partial}{\partial r}\log\theta_y(x)\right)2r(x)\right]$$

$$+ \frac{\mu^2}{2}\Delta \log C_y(x)$$

$$= -\frac{1}{2(\lambda+t-s)}\left[n + r(x)\frac{\partial}{\partial r}\log\theta_y(x)\right] + \frac{\mu^2}{2}\Delta\log C_y(x). \tag{1.14}$$

It is easily shown (see for example [38]) that

$$\langle \nabla r(x), b(x) + \nabla \log B_y(x)\rangle = 0 \tag{1.15}$$

and so,

$$\langle A_s^\lambda(x), b(x)\rangle = -\frac{r(x)}{\lambda+t-s}\langle \nabla r(x), b(x)\rangle$$

$$+ \mu^2\langle \nabla \log C_y(x), b(x)\rangle \tag{1.16}$$

$$= -\frac{r(x)}{\lambda+t-s}\langle \nabla r(x), b(x) + \nabla\log B_y(x)\rangle + \frac{r(x)}{\lambda+t-s}\langle \nabla r(x), \nabla \log B_y(x)\rangle$$

$$+ \mu^2\langle \nabla \log C_y(x), b(x)\rangle$$

$$= \frac{r(x)}{\lambda+t-s} \langle \nabla r(x), \nabla \log B_y(x) \rangle + \mu^2 \langle \nabla \log C_y(x), b(x) \rangle$$

by (1.15)

$$= \frac{r(x)}{\lambda+t-s} \frac{\partial}{\partial r} \log B_y(x) + \mu^2 \langle \nabla \log C_y(x), b(x) \rangle \tag{1.17}$$

By (1.14) and (1.17), we have:

$$\langle A_s^{\lambda,\mu}(x), b(x) \rangle + \frac{1}{2} \Delta Y_s^\lambda(x)$$

$$= -\frac{n}{2(\lambda+t-s)} + \frac{r(x)}{\lambda+t-s} \frac{\partial}{\partial r} \log C_y(x) + \mu^2 (\tfrac{1}{2}\Delta + b) \log C_y(x) \tag{1.18}$$

Finally

$$\frac{1}{2} \frac{\partial Y_s}{\partial s}(x) = -\frac{r^2(x)}{2\mu^2(\lambda+t-s)^2}$$

and so M_t^λ in (1.12) becomes:

$$M_t^\lambda = \exp\{-\frac{r^2(x)}{2\mu^2(\lambda+t)} + \frac{r^2(x_t^{\lambda,\mu})}{2\lambda\mu^2} + \log C_y(x) - \log C_y(x_t^{\lambda,\mu})$$

$$+ \int_0^t (\frac{\mu^2}{2} \|\nabla \log C(x_s^{\lambda,\mu})\|^2 + \mu^2(\tfrac{1}{2}\Delta + b) \log C_y(x_s^{\lambda,\mu}) - \frac{n}{2(\lambda+t-s)}) ds\} \tag{1.19}$$

A direct computation shows that:

$$\frac{1}{2} \|\nabla \log C_y(x)\|^2 + \frac{1}{2} \Delta \log C_y(x) = \frac{1}{2} \frac{\Delta C_y(x)}{C_y(x)} \quad \text{and}$$

$$\langle b(x), \nabla \log C_y(x) \rangle = \langle b(x), \frac{\nabla C_y(x)}{C_y(x)} \rangle .$$

Hence

$$\frac{\mu^2}{2}\|\nabla\log C_y(x)\|^2 + \mu^2(\tfrac{1}{2}\Delta+b)\log C_y(x) = \mu^2\frac{(L-V)C_y(x)}{C_y(x)} \quad . \tag{1.20}$$

Therefore (1.19) becomes:

$$M_t^\lambda = \left(\frac{\lambda}{\lambda+t}\right)^{n/2} \exp\{-\frac{r^2(x)}{2\mu^2(\lambda+t)} + \frac{r^2(x_t^{\lambda,\mu})}{2\lambda\mu^2}\}\frac{C_y(x)}{C_y(x_t^{\lambda,\mu})}\exp\{\mu^2\int_0^t \frac{(L-V)C_y(x_s^{\lambda,\mu})}{C_y(x_s^{\lambda,\mu})}ds\} \tag{1.21}$$

and so the lemma is proved.

The following lemma is a special case of Lemma (3.3) of Chapter II. It is also proved directly in [18].

(1.3) Lemma

$(x_s^{\lambda,\mu})0 \le s \le t$ converges uniformly on $[0,t]$ in probability to a process $(x_s^\mu)0 \le s \le t$ where:

$$x_s^\mu = x_s^{0,\mu} \ \forall\ s \in [0,t)$$

$$x_t^\mu = y \quad \text{a.s.}$$

(1.4) Lemma

The indicator function $\chi_{\tau^{\lambda,\mu}>t}$ of the first exit time $\tau^{\lambda,\mu}$ of the Bridge process $(x_s^{\lambda,\mu})\ 0 \le s \le t \wedge \zeta^\lambda$ from U has the property:

$$\chi_{\tau^{\lambda,\mu}>t} = \lim_p \downarrow \exp\{\int_0^t W_p(x_s^{\lambda,\mu})ds\} \text{ a.s.}$$

where $(W_p)_{p\geq 1}$ is a certain decreasing sequence of bounded continuous functions on M.

Proof. (by Construction)

Define the sequence of open sets:

$$U_p = \{x \in M\colon d(x,\bar{U}) < \tfrac{1}{p}\};\ p \geq 1 \tag{1.23}$$

and define the sequence $(W_p)_{p\geq 1}$ on M as follows:

$$W_p = \begin{cases} 0 \text{ on } \bar{U} \\ -\min(p^2\, d(-,\bar{U}),p) \text{ on } U_p \setminus U \\ -p \text{ on } U_p^c\ . \end{cases} \tag{1.24}$$

Then clearly $(W_p)_{p\geq 1}$ is a decreasing sequence of continuous functions on M each of which is bounded (in fact it is uniformly bounded by 0). Moreover we have:

$$\chi_{\bar{\tau}^{\lambda,\mu}\geq t} = \lim_p \downarrow \exp\{\int_0^t W_p(x_s^{\lambda,\mu})ds\} \tag{1.25}$$

where $\bar{\tau}^{\lambda,\mu}$ is the first time that the Bridge process $(x_s^{\lambda,\mu})$ exits from $\bar{U}$. Since U has smooth boundary,

$$\bar{\tau}^{\lambda,\mu} = \tau^{\lambda,\mu} \text{ a.s.} \tag{1.26}$$

and hence

$$\chi_{\tau^{\lambda,\mu}\geq t} = \lim_{p} \downarrow \exp \{\int_0^t W_p(x_s^{\lambda,\mu})ds\} \text{ a.s.} \tag{1.27}$$

Now, by [19], Theorem 5.2 p. 147, the Cauchy Problem in U for the parabolic equation:

$$\frac{\partial g_s^{\lambda,\mu}}{\partial s} = \frac{\mu^2}{2} \Delta g_s^{\lambda,\mu} + b(g_s^{\lambda,\mu}) + A_s^{\lambda,\mu} (g_s^{\lambda,\mu})$$

$$g_0^{\lambda,\mu} \equiv 1 \tag{1.28}$$

$$g_s^{\lambda,\mu} = 0 \text{ on the boundary } \partial U \text{ of } U$$

has a $C^{1,2}(U)$-solution given by:

$$g_s^{\lambda,\mu}(x) = P_x(\tau^{\lambda,\mu}>s)$$

and hence the measure $P_x(\tau^{\lambda,\mu} \in \cdot)$ has a density with respect to Lebesgue measure on [0,t] and so:

$$\chi_{\tau^{\lambda,\mu}\geq t} = \chi_{\tau^{\lambda,\mu}>t} \quad \text{a.s.} \tag{1.29}$$

The lemma then follows by (1.27).

(1.5(a)) Theorem

If U has compact closure and smooth boundary such that $\bar{U} \subset M \setminus \mathrm{Cut}(y)$ is star-shaped from y, then we have the inequality:

$$p_t^{U,\mu}(x,y) \le (2\pi\mu^2 t)^{-n/2} \exp\{-\frac{d(x,y)^2}{2\mu^2 t}\} C_y(x)$$

$$\times \ \bar{E}_x(\chi_{\tau^\mu > t} \exp\{\mu^2 \int_0^t \frac{LC_y}{C_y}(x_s^\mu)ds\})$$

where $(x_s^\mu)_{0 \le s \le t \wedge \zeta^\mu}$ is the Semi-Classical Brownian Riemannian Bridge with differential generator $\frac{1}{2}\Delta + b + A_s^\mu$ where

$$A_s^\mu(x) = A_s^{0,\mu}(x) = -\frac{x}{A-s} + \mu^2 \nabla \log C_y(x)$$

in geodesic normal coordinates. ζ^μ is the hitting time of the Cut-locus of M at y by the above Bridge Process and τ^μ is its exit time from U.

Proof.

$$p_t^{U,\mu}(x,y) = \lim_{\lambda \downarrow 0} E_x(\chi_{\tau^{\lambda,\mu} > t} f(x_t^{\lambda,\mu}) M_t^\lambda \exp\{\mu^2 \int_0^t V(x_s^{\lambda,\mu})ds\}) \tag{1.30}$$

by (1.7) and Theorem (1.1)

$$= (2\pi\mu^2 t)^{-n/2} \exp\{-\frac{d(x,y)^2}{2\mu^2 t}\} C_y(x) \tag{1.31}$$

$$\times \lim_{\lambda \downarrow 0} E_x(\chi_{\tau^{\lambda,\mu} > t} T_0(x_t^{\lambda,\mu}) C_y^{-1}(x_t^{\lambda,\mu}) \exp\{\mu^2 \int_0^t \frac{LC_y}{C_y}(x_s^{\lambda,\mu})ds\})$$

by Lemma (1.2).

The function C_y (resp. $\frac{LC_y}{C_y}$) is defined and continuous in $\bar{U}$ and hence can be continuously extended to all of M such that the extension be equal to C_y (resp. $\frac{LC_y}{C_y}$) in $\bar{U}$ and 0 outside a bounded neighbourhood of $\bar{U}$. We will denote the extension simply by C_y (resp. $\frac{LC_y}{C_y}$). Hence the expression:

$$T_o(x_t^{\lambda,\mu})C_y^{-1}(x_t^{\lambda,\mu})\exp\{\mu^2\int_0^t(-\frac{LC_y}{C_y}(x_s^{\lambda,\mu}) + \frac{1}{\mu^2}W_p(x_s^{\lambda,\mu}))ds\}$$

is bounded by a constant $C(\mu,p,t)$. By Lemma (1.3), $(x_s^{\lambda,\mu})0 \le s \le t \wedge \zeta^{\lambda,\mu}$ converges (uniformly on $[0,t]$) in probability to $(x_s^{\mu})0 \le s \le t \wedge \zeta^{\mu}$

Set

$$I^{\lambda} = E_x(\chi_{\tau^{\lambda,\mu}>t}\ T_o(x_t^{\lambda,\mu})C_y^{-1}(x_t^{\lambda,\mu})\exp\{\mu^2\int_0^t\frac{LC_y}{Cy}(x_s^{\lambda,\mu})ds\}).$$

By Lemma (1.4) we have:

$$I^{\lambda} \le E_x(T_o(x_t^{\lambda,\mu})C_y(x_t^{\lambda,\mu})\exp\{\mu^2\int_0^t(\frac{LC_y}{C_y}(x_s^{\lambda,\mu})+\frac{1}{\mu^2}W_p(x_s^{\lambda,\mu}))ds\}).$$

Thus taking limits as $\lambda \downarrow 0$ and using Lemma (1.3), we have:

$$\lim_{\lambda\downarrow 0} I^{\lambda} \le E_x(T_o(x_t^{\mu})C_y^{-1}(x_t^{\mu})\exp\{\mu^2\int_0^t(-\frac{LC_y}{C_y}(x_s^{\mu}) + \frac{1}{\mu^2}W_p(x_s^{\mu}))ds\})$$

$x_t^{\mu} = y$ a.s. by Lemma (1.3).

Now, $C_y(y) = 1$ and since we take

$T_o(y) = 1$, the inequality

above becomes:

$$\lim_{\lambda\downarrow 0} I^\lambda \le E_x(\exp\{\mu^2 \int_0^t (\frac{LC_y}{C_y}(x_s^\mu) + \frac{1}{\mu^2} W_p(x_s^\mu))ds\}).$$

By taking limits as $p \uparrow \infty$, we have by using Lemma (1.4) again:

$$\lim_{\lambda\downarrow 0} I^\lambda \le E_x(x_{\tau^\mu > t} \exp\{\mu^2 \int_0^t \frac{LC_y}{C_y}(x_s^\mu)ds\}) \tag{1.32}$$

and so we obtain the inequality of the Theorem.

(1.5(b)) <u>Theorem</u>

The reverse inequality of Theorem (1.5(a)) is also valid under the same hypotheses:

$$p_t^{U,\mu}(x,y) \ge (2\pi\mu^2 t)^{-n/2} \exp\{-\frac{d(x,y)^2}{2\mu^2 t}\} C_y(x)$$

$$\times E_x(x_{\tau^\mu > t} \exp\{\mu^2 \int_0^t \frac{LC_y}{C_y}(x_s^\mu)ds\}).$$

<u>Proof</u>.

The proof follows that of Theorem 2B in [18]: We know by (1.7) and Theorem (1.1) that:

$$p_t^{U,\mu}(x,y) = \lim_{\lambda\downarrow 0} E_x(x_{\tau^{\lambda,\mu} > t} f_0^\lambda(x_t^{\lambda,\mu}) M_t^\lambda \exp\{\mu^2 \int_0^t V(x_s^{\lambda,\mu})ds\}) \tag{1.33}$$

$$= (2\pi\mu^2 t)^{-n/2} \exp\{-\frac{d(x,y)^2}{2\mu^2 t}\} C_y(x) \tag{1.34}$$

$$\times \lim_{\lambda\downarrow 0} E_x(x_{\tau^{\lambda,\mu} > t} T_0(x_t^{\lambda,\mu}) C_y^{-1}(x_t^{\lambda,\mu}) \exp\{\mu^2 \int_0^t \frac{LC_y}{C_y}(x_s^{\lambda,\mu})ds\})$$

$$= (2\pi\mu^2 t)^{-n/2} \exp\{-\frac{d(x,y)^2}{2\mu^2 t}\} C_y(x) \lim_{\lambda\downarrow 0} I^\lambda . \tag{1.35}$$

Now, let Γ be the space of continuous paths $\sigma\colon [0,t] \to M^+$ where M^+ is the one-point compactification of M with $\sigma(0) = x \in U$.

Let $\tau\colon\Gamma \to \mathbb{R} \cup \{+\infty\}$ be the map defined by: $\tau(\sigma)$ is the first exit time of σ from U. Then, (1.35) becomes:

$$\lim_{\lambda\downarrow 0} \int_{\hat{\Gamma}} \chi_{\{\tau>t\}}(\sigma)\, T_{\sigma}(\sigma(t))\, C_y^{-1}(\sigma(t)) \qquad (1.36)$$
$$\times \exp\{\mu^2 \int_0^t \frac{LC_y}{C_y}(\sigma(s))ds\} dP_x^{\lambda,\mu}(\sigma)$$

where $P_x^{\lambda,\mu} = x_\cdot^{\lambda,\mu}(P_x)$ is the image measure of P_x by $x_\cdot^{\lambda,\mu}$.

We know that convergence in probability implies convergence in law. Hence, since $x_\cdot^{\lambda,\mu}$ converges in probability to $x_\cdot^{\mu}$ by Lemma (1.3), we conclude that $P_x^{\lambda,\mu}$ converges weakly (or narrowly) to $P_x^{\mu} = x_\cdot^{\mu}(P_x)$. Consequently, by ([35], Appendix, Proposition 1) we have the inequality of the Theorem.

(1.6) <u>Theorem</u>

For $\bar{U} \subset M \setminus Cut(y)$ star-shaped from y with compact closure and smooth boundary,

$$p_t^{U,\mu}(x,y) = (2\pi\mu^2 t)^{-n/2} \exp\{-\frac{d(x,y)^2}{2\mu^2 t}\}\, C_y(x)$$
$$\times\ E_x(\chi_{\tau^\mu>t} \exp\{\mu^2 \int_0^t \frac{LC_y}{C_y}(x_s^\mu)ds\}).$$

<u>Proof</u>

The result is immediate by the inequality of Theorem (1.5(a)) and the reverse inequality proved in Theorem (1.5(b)).

(1.7) <u>Corollary</u>

(i) For any $U \subset M \setminus \mathrm{Cut}(y)$ star-shaped from y, we have:

$$p_t^{U,\mu}(x,y) = (2\pi\mu^2 t)^{-n/2} \exp\{-\frac{d(x,y)^2}{2\mu^2 t}\} C_y(x)$$

$$\times\ E_x(\chi_{\tau^\mu > t} \exp\{\mu^2 \int_0^t \frac{LC_y}{C_y}(x_s^\mu)ds\})$$

(ii) $$p_t^{M,\mu}(x,y) = (2\pi\mu^2 t)^{-n/2} \exp\{-\frac{d(x,y)^2}{2\mu^2 t}\} C_y(x)$$

$$\times\ E_x(\exp\{\mu^2 \int_0^t \frac{LC_y}{C_y}(x_s^\mu)ds\})$$

when M has a pole at y.

In particular, the expectation in (ii) is finite.

<u>Proof</u>.

(i) Let $(U_k)_{k\geq 1}$ be an increasing sequence of open subsets of U exhausting U such that each U_k has compact closure with smooth boundary and each $\bar{U}_k \subset M \setminus \mathrm{Cut}(y)$. Then for each $k \geq 1$, $p_t^{U_k,\mu}(x,y)$ satisfies the equality of Theorem (1.6). Then taking limits as $k \uparrow \infty$, and using the monotone convergence Theorem, we have:

$$\lim_k \uparrow p_t^{U_k,\mu}(x,y) = (2\pi\mu t)^{-n/2} \exp\{-\frac{d(x,y)^2}{2\mu^2 t}\} C_y(x)$$

$$\times\ E_x(\chi_{\tau^\mu > t} \exp\{\mu^2 \int_0^t \frac{LC_y}{C_y}(x_s^\mu)ds\}), \tag{1.37}$$

since $\tau_k^\mu \uparrow \tau^\mu$.

Now, $\lim_k \uparrow p_t^{U_k,\mu}(x,y) = p_t^{U,\mu}(x,y)$ (1.38)

by ([11], Chapter VIII, Theorem 4) and so (i) is proved.

(ii) is immediate by taking $U = M$ in (i) since $\mathrm{Cut}(y) = \phi$ in this case.

We will now proceed to give some applications of the Heat Kernel Formula we have just obtained in Theorem (1.6).

§2. SOME APPLICATIONS

We will first consider the Heat Kernel Formula for the standard n-sphere $S^n = (S^n(1), g_0)$ for $n \geq 2$.

Fix a point $y \in S^n$ and let $\bar{y}$ be the point anti-podal to y. Let $B(\bar{y},\varepsilon)$ for small $\varepsilon > 0$ be the geodesic ball with centre $\bar{y}$ and radius ε.

Let $C_\varepsilon = S^n \setminus B(\bar{y},\varepsilon)$ and let $q_t^\varepsilon(-,-)$ be the Dirichlet Heat Kernel of C_ε. By ([12], p. 7) we have:

$$p_t^{S^n}(-,0) = \lim_{\varepsilon \downarrow 0} q_t^\varepsilon(-,-) \tag{2.1}$$

on $(S^n \setminus \{\bar{y}\}) \times (S^n \setminus \{\bar{y}\}) \times (0,\infty)$.

Now, $q_t^\varepsilon(-,-)$ is also the Dirichlet Heat Kernel for the interior $\mathring{C}_\varepsilon$ of C_ε and hence:

$$p_t^{S^n}(x,y) = \lim_{\varepsilon \downarrow 0} p_t^{\mathring{C}_\varepsilon}(x,y) \tag{2.2}$$

for all $x \neq \bar{y}$. Now, by Theorem (1.6),

$$p_t^{\mathring{C}_\varepsilon}(x,y) = (2\pi t)^{-n/2} C_y(x) \exp\{-\frac{d(x,y)^2}{2t}\} \qquad (2.3)$$

$$\times E_x(\chi_{\tau^\varepsilon > t} \exp\{\int_0^t -\frac{LC_y}{C_y}(x_s^t)ds\})$$

where τ^ε is the first time that the Bridge Process $(x_s^t)_{0 \le s \le t \wedge \zeta}$ exits from $\mathring{C}_\varepsilon$ and ζ is the first hitting time of the anti-podal point $\bar{y}$ by the above Bridge Process started at x. Since $\tau^\varepsilon \uparrow \zeta$ as $\varepsilon \downarrow 0$, we have

$$p_t^{S^n}(x,y) = (2\pi t)^{-n/2} C_y(x) \exp\{-\frac{d(x,y)^2}{2t}\} \qquad (2.4)$$

$$\times E_x(\chi_{\zeta > t} \exp\{\int_0^t -\frac{LC_y}{C_y}(x_s^t)ds\})$$

for all $x \neq \bar{y}$.

We will take $b \equiv 0$ and $V \equiv 0$ and have:

$$p_t^{S^n}(x,y) = (2\pi t)^{-n/2} \theta_y^{-\frac{1}{2}}(x) \exp\{-\frac{d(x,y)^2}{2t}\} \qquad (2.5)$$

$$\times E_x(\chi_{\zeta > t} \exp\{\int_0^t \frac{1}{2} \theta_y^{\frac{1}{2}}(x_s^t) \Delta \theta_y^{-\frac{1}{2}}(x_s^t)ds\})$$

Set $r = d(x,y)$, then

$$\theta_y(x) = (\frac{\sin r}{r})^{n-1} \quad \text{for } 0 < r < \pi \quad \text{and}$$

$$\frac{1}{2} \theta_y^{\frac{1}{2}}(x) \Delta \theta_y^{-\frac{1}{2}}(x) = \frac{(n-1)^2}{8} + \frac{(n-1)(n-3)}{8} (\frac{1}{r^2} - \frac{1}{\sin^2 r}) \qquad (2.6)$$

for $n \ge 2$. In particular for $n = 3$,

$$p_t^{S^3}(x,y) = (2\pi t)^{-3/2} \frac{r}{\sin r} e^{-\frac{r^2}{2t}} e^{\frac{t}{2}} P_x(\zeta > t). \tag{2.7}$$

The radial component of the Bridge $(x_s^t)0 \le s \le t \wedge \zeta$ has the same distribution as the radial component of the corresponding Euclidean Brownian Bridge $(B_s^t)0 \le s \le t \wedge \zeta_o$ from x to 0 in time t by ([14], Chapter IX, proof of Theorem 12C(i)).

Hence we have:

$$P_x(\zeta > t) = \bar{P}_x(\zeta_o > t) \tag{2.8}$$

where $\zeta_o = \zeta_o(t)$ is the first exit time from the Euclidean ball $D = D(0,\pi)$ of the n-dimensional Euclidean Brownian Bridge $(B_s^t)0 \le s \le t \wedge \zeta_o$. Thus by (2.7), we have:

$$p_t^{S^3}(x,y) = (2\pi t)^{-3/2} \frac{r}{\sin r} e^{-\frac{r^2}{2t}} e^{t/2} P_x(\zeta_o > t). \tag{2.9}$$

By Theorem (1.6), we have for $M = R^3$, $U = D$:

$$p_t^D(x,0) = (2\pi t)^{-3/2} \; e^{-(r^2/2t)} P_x(\zeta_o > t). \tag{2.10}$$

We next consider the eigen problem in D:

$$\begin{aligned} \Delta\phi + \lambda\phi &= 0 \\ \phi|_{\partial D} &\equiv 0 \end{aligned} \tag{2.11}$$

Then by ([13], Chapter V, §8),

$$p_t^D(x,0) = \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{n,p} t} \phi_{n,p}(x) \cdot \phi_{n,p}(0) \tag{2.12}$$

where:

(i) $\phi_{n,p}(x) = Y_n(\theta,\phi) \cdot S_n(r.\sqrt{\lambda}_{n,p})$

where $x \to (\theta,\phi,r)$ is the change from Cartesian to spherical coordinates.

(ii) $S_n(r.\sqrt{\lambda}) = \dfrac{J_{n+\frac{1}{2}}(r.\sqrt{\lambda})}{r}$

(which is regular at $r = 0$), J_n being the Bessel function of order n and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,p}, \ldots)$ are solutions of the equation:

$$J_{n+\frac{1}{2}}(\sqrt{\lambda}) = 0.$$

By (2.9), (2.10) and (2.12),

$$p_t^{S^3}(x,y) = \frac{r}{\sin r} e^{t/2} p_t^D(x,0) \tag{2.13}$$

$$= \frac{r}{\sin r} e^{t/2} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{n,p} t} . \phi_{n,p}(x) . \phi_{n,p}(0). \tag{2.14}$$

On the other hand we have the eigenproblem in S^3:

$$\Delta\phi + \mu\phi = 0 \tag{2.15}$$

and hence,

$$p_t^{S^3}(x,y) = \sum_{k=0}^{\infty} \phi_k(x)\phi_k(y)e^{-\mu_k t} \tag{2.16}$$

where by ([7]; Chapter III; Prop. C.I.1).

(i) $\mu_k = k(k+2)$

(ii) ϕ_k is a homogeneous polynomial of degree k harmonic on $\mathbb{R}^4$.

Lastly we have a third formula recently proved by K.D. Elworthy in [15] by using (2.5) above and the "method of images" (as a special case of the formula for Compact Lie Groups in [29]):

$$p_t^{S^3}(x,y) = (2\pi t)^{-3/2}\, e^{t/2} \sum_{\gamma} \frac{\ell(\gamma)}{\sin(\ell(\gamma))} \exp\{-\frac{\ell(\gamma)^2}{2t}\} \tag{2.17}$$

where the sum is taken over all geodesics γ from x to y and $\ell(\gamma)$ is the length of γ. These lengths are given by:

$$\gamma_k = 2\pi k + r;\quad k = 0,1,2,\ldots$$

$$\gamma_k = 2\pi k - r;\quad k = 1,2,3,\ldots$$

for $0 < r < \pi$.

Thus (2.17) becomes:

$$p_t^{S^3}(x,y) = (2\pi t)^{-3/2}\, e^{t/2}\Big[\sum_{k=0}^{\infty} \frac{(2\pi k+r)}{\sin(2\pi k+r)}\, e^{-\frac{(2\pi k+r)^2}{2t}} + \sum_{k=1}^{\infty} \frac{(2\pi k-r)}{\sin(2\pi k-r)}\, e^{-\frac{(2\pi k-r)^2}{2t}}\Big] \tag{2.18}$$

$$= (2\pi t)^{-3/2} e^{t/2} e^{-\frac{r^2}{2t}} \frac{r}{\sin r} [1-2 \sum_{k=1}^{\infty} (\frac{2\pi k}{r} \sinh(\frac{2\pi kr}{t}) - \cosh(\frac{2\pi kr}{t})) e^{-\frac{2\pi^2 k^2}{t}}] \tag{2.19}$$

Consequently by (2.14), (2.16) and (2.19), we have the identities:

$$\frac{r}{\sin r} e^{t/2} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{n,p} t} \phi_{n,p}(x) \phi_{n,p}(0) \tag{2.20}$$

$$= \sum_{k=0}^{\infty} e^{-k(k+2)t} \phi_k(x)\phi_k(y) \tag{2.21}$$

$$= (2\pi t)^{-3/2} e^{t/2} e^{-\frac{r^2}{2t}} \frac{r}{\sin r} [1-2 \sum_{k=1}^{\infty} (\frac{2\pi k}{r} \sinh(\frac{2\pi kr}{t}) - \cosh(\frac{2\pi kr}{t})) e^{-\frac{2\pi^2 k^2}{t}}].$$

We will now derive some consequences of some of the formulae we have just obtained:

(i) As a direct consequence of (2.7) and (2.19), we have:

$$P_x(\zeta > t) = 1-2 \sum_{k=1}^{\infty} e^{-\frac{2\pi^2 k^2}{t}} (\frac{2\pi k}{r} \sinh(\frac{2\pi kr}{t}) - \cosh(\frac{2\pi kr}{t})). \tag{2.22}$$

By (2.8), $P_x(\sup_{0 \le s \le t} |B_s^t| < \pi)$ is equal to the R.H.S. of (2.22) above. Thus by computing on the sphere $S^3(\frac{\rho}{\pi})$ instead of $S^3(1)$, we have:

$$P_x(\sup_{0 \le s \le t} |B_s^t| < \rho) = 1-2 \sum_{k=1}^{\infty} e^{-\frac{2\rho^2 k^2}{t}} (\frac{2\rho k}{r} \sinh(\frac{2\pi kr}{t}) - \cosh(\frac{2\pi kr}{t})) \tag{2.23}$$

for $0 < r < \rho$. Now, let $t = 1$ and take limits as $r \downarrow 0$ in (2.23) above; then:

$$P_0\left(\sup_{0\le s\le 1} |B_s^1| < \rho\right) = 1-2\sum_{k=1}^{\infty} e^{-2\rho^2k^2}(4\rho^2k^2-1). \tag{2.24}$$

On the other hand, by (1.1) of Theorem 1 in [26]

$$P_0\left(\sup_{0\le s\le 1} w(s) < \rho\right) = 1-2\sum_{k=1}^{\infty} e^{-2\rho^2k^2}(4\rho^2k^2-1) \tag{2.25}$$

where $(w(s))0 \le s \le 1$ is the unsigned scaled Brownian excursion (see for example [26] for definition). The above results in (2.24) and (2.25) confirm D. Williams observation in [39] that $(w(s))0 \le s \le 1$ has the same distribution as the 3-dimensional Bessel Bridge $|B_s^1|$ $0 \le s \le 1$. Thus, (2.23) generalizes (1.1) of Theorem 1 in [26].

(ii) By (2.19), we have:

$$p_t^{S^3}(y,y) = (2\pi t)^{-3/2} e^{t/2}\left[1-2\sum_{k=1}^{\infty} e^{-\frac{2\pi^2k^2}{t}}(4\pi^2k^2-1)\right] \tag{2.26}$$

Since $-2\sum_{k=1}^{\infty} e^{-\frac{2\pi^2k^2}{t}}(4\pi^2k^2-1) = o(t^n)$ $\forall n \ge 1$, we have:

$$(2\pi t)^{3/2}\, p_t^{S^3}(y,y) = e^{t/2} + o(t^n) \quad \forall n \ge 1 \tag{2.27}$$

$$= 1 + \frac{1}{2}t + \frac{1}{2!}\left(\frac{1}{2}\right)^2t^2 + \dots + \frac{1}{n!}\left(\frac{1}{2}\right)^nt^n + o(t^n) \quad \forall n \ge 1 \tag{2.28}$$

and so we obtain all the terms of the expansion to any order of H.P. McKean and I.M. Singer for S^3 given in [30].

The expansion above in (2.28) will be generalized later to more general manifolds and a more general operator after obtaining asymptotics of the Heat Kernel in a geodesic chart.

An extra result of the Heat Kernel Formula for $\mathbb{S}^n$ is the Heat Kernel Formula for the Real Projective Space $\mathbb{RP}^n$. Consider the covering map:

$$\pi : \mathbb{S}^n \longrightarrow \mathbb{RP}^n.$$

Let $\bar{x} = -x$ be the point in $\mathbb{S}^n$ antipodal to x. Fix a point $y \in \mathbb{S}^n$. Then set $\dot{x} = \pi(x) = \pi(\bar{x})$

$$\dot{y} = \pi(y) = \pi(\bar{y}).$$

Then we have by ([14]; Chap. IX, Remark 12D(i))

$$p_t^{\mathbb{RP}^n}(\dot{x},\dot{y}) = \sum_{z\in\pi^{-1}(\dot{x})} p_t^{\mathbb{S}^n}(z,y) = \sum_{z\in\pi^{-1}(\dot{x})} p_t^{\mathbb{S}^n}(z,\bar{y}) \tag{2.29}$$

$$= p_t^{\mathbb{S}^n}(x,y) + p_t^{\mathbb{S}^n}(\bar{x},y) = p_t^{\mathbb{S}^n}(x,\bar{y}) + p_t^{\mathbb{S}^n}(\bar{x},\bar{y}). \tag{2.30}$$

We thus obtain a stochastic representation of the Heat Kernel of $\mathbb{RP}^n$ by knowledge of that of $\mathbb{S}^n$ given in (2.4).

Note that in (2.30) above, x and $\bar{x} \neq \bar{y}$ in the first equality and x and $\bar{x} \neq y$ in the second equality.

CHAPTER II: AN INTEGRAL FORMULA FOR THE HEAT KERNEL

§0. INTRODUCTION

This Chapter and the next seek to generalize the results of Chapter I. We generalize the point y to a submanifold N of M and the exponential chart $(\exp_y^{-1}, U)$ to a chart $(\exp_\nu^{-1}, M_o)$ of the exponential map of the normal bundle of N. Finally the geodesic normal coordinates are generalized to the so-called Fermi coordinates which we define below.

§1. FERMI COORDINATES

We need Fermi coordinates to describe the geometry of a Riemannian manifold in the neighbourhood of a submanifold (see [20] where they are defined and extensively used).

Let M be a Riemannian manifold of dimension n and N a submanifold of dimension k; $0 \le k \le n-1$. Let $\nu: E_\nu \to N$ be the normal bundle of N and let $\exp_\nu : E_\nu \to M$ be its exponential map. Then $\exp_\nu$ maps a neighbourhood E_ν^o of the zero section of ν diffeomorphically onto a subset M_o of M. More precisely, E_ν^o is defined as follows. Let S(N) denote the sphere (sub) bundle associated to ν:

$$S(N) = \{(y,\xi) \in E_\nu : \ \|\xi\| = 1\}.$$

Define the function $c:S(N) \to \mathbb{R}_+$ as follows:

$$c(y,\xi) = \sup \{\rho \ge 0 : d(\exp_\nu(y,\rho\xi),y) = \rho\}.$$

Then,

$$E_\nu^o = \{(y,\rho\xi) \in E_\nu : 0 \le \rho < c(y,\xi)\}$$

and $M_o = \exp_\nu(E_\nu^o)$. Then M_o contains N since N is the image of the zero section under the exponential map $\exp_\nu$.

(1.1) Definitions

(i) We say that E_ν^o is star-shaped in E_ν.

(ii) M_o is also said to be star-shaped from N in M and is also called a tubular neighbourhood of N in M.

(iii) If, in the definition of E_ν^o, we replace $c(y,\xi)$ by a real number $\rho_o > 0$, then we call M_o a tubular neighbourhood of N of radius ρ_o around N.

Let $y \in N$ and let $E_{k+1},\ldots,E_n$ be orthonormal sections of ν defined in a neighbourhood $V \subset N$ of y. Let $(y_1,\ldots,y_k)$ be a coordinate system on N at y.

(1.2) Definitions

(i) The Cartesian Fermi Coordinates $(x_1,\ldots,x_k,\ldots,x_n)$ of $N \subset M$ at $y \in N$ relative to the coordinate system $(y_1,\ldots,y_k)$ and normal fields $E_{k+1},\ldots,E_n$ are given by:

$$x_\alpha(\exp_\nu(q, \sum_{j=k+1}^{n} t_j E_j(q))) = y_\alpha(q) \text{ for } \alpha = 1,\dots,k$$

$$x_\alpha(\exp_\nu(q, \sum_{j=k+1}^{n} t_j E_j(q))) = t_\alpha \text{ for } \alpha = k+1,\dots,n$$

for $q \in V$. Thus when $N = \{y\}$, the Fermi coordinates reduce to ordinary normal coordinates at y.

(ii) Set $\rho^2(x) = \sum_{j=k+1}^{n} x_j^2(x)$.

Then by (Lemma (2.1) of [20]),

$$d(x,N) = \rho(x) = (\sum_{j=k+1}^{n} x_j^2(x))^{\frac{1}{2}}.$$

The tubular neighbourhood M_o of N is characterized by (see §2 and §4 of [20]) the existence, for each $x \in M_o$, of a unique minimal geodesic of unit speed:

$$\sigma_N : [0,t] \to M$$

from $y \in N$ to x and meeting N orthogonally i.e.

$$\sigma_N(0) = y \in N; \quad \sigma_N(t) = x$$

$$|\sigma_N'(s)| = 1; \qquad \sigma_N'(0) \in (T_yN)^{\perp}.$$

Minimizing above means minimizing distance from N to $x \in M_o$. Now, let

$$\alpha_N : [0,1] \to M$$

be defined by:

$$\alpha_N(s) = \sigma_N(ts).$$

Then α_N is a unique minimal geodesic from $y \in N$ to $x \in M_o$ in time 1. Finally define

(iii) $\gamma_N : [0,1] \to M$

by $\gamma_N(s) = \alpha_N(1-s)$. Then γ_N is the unique minimal geodesic from $x \in M_o$ to $y \in N$ in time 1.

(iv) Define $B_N : M_o \to \mathbb{R}$ by

$$B_N(x) = \exp\left\{\int_0^1 \langle\gamma_N'(s), b(\gamma_N(s))\rangle_{\gamma_N(s)} ds\right\}$$

where b is a smooth vector field on M.

Let θ_N be the volume change factor under the exponential map: $\exp_\nu : E_\nu^o \to M_o$. By definition, assuming that $(y_1,\dots,y_k)$ are normal at $y \in N$,

(v) $\theta_N(y,v) = \sqrt{g(y,v)} = (\det g_{ij}(y,v))^{\frac{1}{2}}$

where $g_{ij} = \langle\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\rangle$ and $(x_1,\dots,x_k,\dots,x_n)$ are the Cartesian Fermi Coordinates defined above.

Recalling that:

$$\rho = d(-,N) = (\sum_{j=k+1}^{n} x_j^2)^{\frac{1}{2}},$$

(vi) The corresponding polar Fermi coordinates are given by $(x_1,\ldots,x_k,\rho,\xi_{k+2},\ldots,\xi_n)$ where

$$\xi_i = \frac{x_i}{\rho} \; ; \; i = k+2,\ldots,n.$$

(vii) Let (g^{ij}) be the inverse of the matrix

$$(g_{ij}); \; i,j = 1,\ldots,n:$$

$$g^{ij} = \langle dx_i,\, dx_j\rangle.$$

Then by ([22], Theorem 2) we have the Lemma:

(1.3) Lemma

The Laplacian Δ in polar Fermi coordinates is given by:

$$= \frac{\partial^2}{\partial\rho^2} + \left(\frac{n-k-1}{\rho} + \frac{\partial\theta_N}{\partial\rho}/\theta_N\right)\frac{\partial}{\partial\rho} + \frac{1}{\theta_N}\sum_{\alpha,\beta=1}^{k}\frac{\partial}{\partial x_\alpha}\left(g^{\alpha\beta}\theta_N\frac{\partial}{\partial x_\beta}\right)$$

$$+ \frac{1}{\rho\theta_N}\sum_{\alpha,j}\frac{\partial}{\partial x_\alpha}\left(g^{\alpha j}\theta_N\frac{\partial}{\partial\xi_j}\right) + \frac{1}{\rho^2\theta_N}\sum_{ij}\frac{\partial}{\partial\xi_i}\left(g^{ij}\theta_N\frac{\partial}{\partial\xi_j}\right)$$

where $\alpha,\beta = 1,\ldots,k$; $i,j = k+2,\ldots,n$. The above is given along the geodesic $\xi_{k+2} = \ldots = \xi_n = 0$.

§2. SOME AUXILIARY RESULTS OF RIEMANNIAN GEOMETRY

The polar Fermi coordinates $(x_1,\dots,x_k,\rho,\xi_{k+2},\dots,\xi_n)$ induce a metric in the neighbourhood M_0 of N as follows:

$$(ds_N)^2 = (d\rho)^2 + g_{\alpha\beta}dx_\alpha\, dx_\beta + g_{\alpha j}\, dx_\alpha\, d\xi_j + g_{ij}\, d\xi_i d\xi_j \tag{2.1}$$

for $\alpha,\beta = 1,\dots,k;\ i,j = k+2,\dots,n$ where

$$g_{\alpha\beta} = \langle\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\rangle;\ g_{\alpha j} = \langle\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_j}\rangle \tag{2.2}$$

$$g_{ij} = \langle\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\rangle . \tag{2.3}$$

We will denote the above metric simply by $g = (g_{ij})$. The diffeomorphism:

$$\exp_\nu : E_\nu^0 \to M_0$$

induces an isometry of E_ν^0 onto M_0. We will denote the corresponding metric on E_ν^0 by f which is defined by:

$$f_{ij}(\omega) = \langle\frac{\partial}{\partial\omega_i}, \frac{\partial}{\partial\omega_j}\rangle^\sim_\omega \tag{2.4}$$

where $\langle\ ,\ \rangle^\sim_\omega$ is the inner product on the fibres of TE_ν defined as follows:

(2.1) Definition

For each $\omega \in E_\nu^0$ and each pair $W_1, W_2 \in T_\omega E_\nu$,

$$\langle W_1, W_2 \rangle^{\sim}_{\omega} = \langle (T_{\omega} \exp_{\nu}) W_1, (T_{\omega} \exp_{\nu}) W_2 \rangle_{\exp_{\nu}(\omega)} \tag{2.5}$$

where $\langle\ ,\ \rangle_x$ is the inner product on T_xM for each $x \in M_0$.

(2.2) Lemma (due to K.D. Elworthy)

There exists a Riemannian manifold $\tilde{M}$ and a submanifold $\tilde{M}_0$ of $\tilde{M}$ which is isometric to M_0 and such that $\tilde{M}$ has no cut-focal points with respect to $\tilde{N} = \exp_{\nu}^{-1}(N)$.

Before we begin the proof, we recall that a cut-focal point (with respect to a submanifold N) along a geodesic γ meeting N orthogonally (i.e. $\gamma'(0) \in (T_{\gamma(0)}N)^{\perp}$) is the first point beyond which there are shorter geodesics that meet N orthogonally.

Proof of Lemma (2.2)

By the connexion on the normal bundle, the tangent bundle to the normal bundle splits:

$$TE_{\nu} = HE_{\nu} \oplus VE_{\nu} \tag{2.6}$$

$$\text{i.e.} \quad \forall\omega \in E_{\omega},\ T_{\omega}E_{\nu} = H_{\omega}E_{\nu} \oplus V_{\omega}E_{\nu} \tag{2.7}$$

where $H_{\omega}E_{\nu}$ = Vector space of Horizontal Vectors and $V_{\omega}E_{\nu}$ = Vector Space of Vertical Vectors.

Let us denote by $H_{\omega}(W)$ (resp. $V_{\omega}(W)$) the horizontal (resp. vertical) component of a vector $W \in T_{\omega}E_{\nu}$; then the natural metric on E_{ν} is defined via the inner product $\langle\ ,\ \rangle_{\omega}$ on each $T_{\omega}E_{\nu}$ which, itself, is defined as follows:

For each pair $W_1, W_2 \in T_\omega E_\nu$,

$$\langle W_1,W_2\rangle_\omega = \langle H_\omega(W_1),H_\omega(W_2)\rangle^1 + \langle V_\omega(W_1),V_\omega(W_2)\rangle^2 \tag{2.8}$$

where $\langle\ ,\ \rangle^1$ (resp. $\langle\ ,\ \rangle^2$) is the inner product of $H_\omega E_\nu \cong T_xN$ (resp. $V_\omega E_\nu \cong (T_xN)^\perp$) for each $x = \nu(\omega)$. Let h denote this metric on E_ν.

Next suppose that $E_\nu^0 \subset \overline{E_\nu^0} \subset \overline{E_\nu^1}$ where $\exp_\nu(E_\nu^1)$ is a tubular neighbourhood of N in M. Then there exists a C^∞ function:

$$\mu: E_\nu \longrightarrow [0,1]$$

such that:

$$\mu|_{\overline{E^0}} \equiv 1 \text{ and } \operatorname{supp}\mu \subset E_\nu^1.$$

Finally define $\ell = \mu f + (1-\mu)h.$ (2.9)

Then ℓ is a metric on E_ν and E_ν with this metric has no cut-focal points with respect to $\tilde{N} = \exp_\nu^{-1}(N)$ since a geodesic γ meeting $\tilde{N}$ orthogonally is just a ray passing through the origin of the fibre $(T_{\gamma(0)}N)^\perp$ of E_ν. Take $\tilde{M}$ to be E_ν with the metric ℓ.

<u>N.B</u>: $\tilde{N}$ above is just the zero section of the normal bundle E_ν.

§3. <u>THE SEMI-CLASSICAL BRIDGE FROM A POINT TO A SUBMANIFOLD</u>

(3.1) <u>Definitions</u>

(i) Define $Y_s^\lambda : M_0 \to R$ by

$$Y_s^\lambda(x) = -\frac{\rho^2(x)}{2(\lambda+t-s)} + \log C_N(x) \tag{3.1}$$

for small $\lambda > 0$ where

$$C_N(x) = B_N(x)\cdot\theta_N^{-\frac{1}{2}}(x) \qquad (3.2)$$

(ii) Set $A_s^\lambda = \nabla Y_s^\lambda$ (3.3)

(iii) We define (up to exit time from the tubular neighbourhood M_0) the Semi-Classical Brownian Riemannian Bridge from $x \in M_0$ to the submanifold N in time $t+\lambda$ as the diffusion process on M with differential operator $L_s^\lambda = \frac{1}{2}\Delta + b + A_s^\lambda$.

The next two Lemmas are on the radial behaviour of the Semi-Classical Bridge Process $(x_s^\lambda)0 \leq s \leq t$. By Lemma (2.2) we will be assuming that $(\exp_\nu^{-1}, M)$ is a global chart. Also, we will assume that N is of compact closure and smooth boundary.

(3.2(a)) Lemma

For $n-k \geq 2$, $\rho^\lambda = \rho(x_s^\lambda) = d(x_s^\lambda, N)$ (3.4)

is equal in distribution to the radial component of the Euclidean Brownian Bridge on R^{n-k} from v to 0 in time $t+\lambda$.

Proof.

First recall that $\rho_s^\lambda = |v_s^\lambda|$ where $x_s^\lambda = \exp_\nu(y_s^\lambda, v_s^\lambda)$. Let ζ^λ be the explosion time of (x_s^λ) and set

$$\zeta_N^\lambda = \inf\{s \geq 0 : \rho_s^\lambda = 0\}. \qquad (3.5)$$

Then ζ_N^λ is the hitting time of N by the Submanifold Bridge Process (x_s^λ) started at $x \in (M \setminus N)$.

Now, ρ is C^2 on $(M \setminus N)$ and hence by Itô's Formula:

$$\rho_s^\lambda = \rho(x) + \int_0^s d\rho(x_r^\lambda dB_r) + \int_0^s \langle\nabla\rho(x_r^\lambda), b(x_r^\lambda) + A_r^\lambda(x_r^\lambda)\rangle dr$$

$$+ \frac{1}{2}\int_0^s \Delta\rho(x^\lambda)dr \text{ a.s. on } \{\zeta^\lambda > t\} \cap \{\zeta_N^\lambda > t\} \tag{3.6}$$

$$\text{Now, } A_r^\lambda(x) = -\frac{\nabla\rho^2(x)}{2(\lambda+t-r)} + \nabla \log C_N(x) \tag{3.7}$$

$$= -\sum_{j=k+1}^{n} \frac{x_j(x)(\frac{\partial}{\partial x_j})x}{\lambda+t-r} + \nabla \log B_N(x) - \frac{1}{2}\nabla \log \theta_N(x). \tag{3.8}$$

We will show that:

$$\langle\nabla\rho(x), b(x) + \nabla \log B_N(x)\rangle = 0 \tag{3.9}$$

Let α_N (resp. γ_N) be the geodesic of Definition (1.2(ii)) (resp. (1.2(iii))): α_N is parameterized to take unit time from $y \in N$ to $x \in (M \setminus N)$ and γ_N is parameterized to take unit time from $x \in (M \setminus N)$ to $y \in N$.

By Lemma (2.1) of [20] and (the Generalized Gauss) Lemma (3.1) of [20],

$$\nabla\rho(\alpha_N(s)) = \alpha_N'(s);\ 0 \le s \le 1 \tag{3.10}$$

$$\text{i.e.} \quad \nabla\rho(\gamma_N(1-s)) = -\gamma_N'(1-s);\ 0 \le s \le 1 \tag{3.11}$$

By definition,

$$\frac{d}{ds} \log B_N(\gamma_N(s))_{|s=0} = \langle \nabla \ \log B_N(\gamma_N(0)), \gamma_N'(0)\rangle \tag{3.12}$$

$$= -\langle \nabla \log B_N(x), \nabla\rho(x)\rangle \tag{3.13}$$

by (2.13).

On the other hand, by the definition of B_N,

$$\log B_N(\gamma_N(s)) = \int_0^1 \langle\beta_N'(r), b(\beta_N(r))\rangle_{\beta_N(r)} dr \tag{3.14}$$

where β_N is the unique minimal goedesic from $\gamma_N(s)$ to y in time 1. Since β_N goes from $\gamma_N(s)$ to y in unit time,

$$\int_0^1 \langle\beta_N'(r), b(\beta_N(r))\rangle_{\beta_N(r)} dr = \int_s^1 \langle\gamma_N'(r), b(\gamma_N(r))\rangle_{\gamma_N(r)} dr \tag{3.15}$$

Hence,

$$\log B_N(\gamma_N(s)) = \int_s^1 \langle\gamma_N'(r), b(\gamma_N(r))\rangle_{\gamma_N(r)} dr \tag{3.16}$$

and so,

$$\frac{d}{ds} \log B_N(\gamma_N(s))_{|s=0} = - \langle\gamma_N'(0), b(\gamma_N(0))\rangle_x \tag{3.17}$$

$$= - \langle\gamma_N'(0), b(x)\rangle_x \tag{3.18}$$

i.e.

$$\frac{d}{ds} \log B_N(\gamma_N(s))_{|s=0} = \langle\nabla\rho(x), b(x)\rangle \tag{3.19}$$

by (3.11). Consequently by (3.19) and (3.13),

$$\langle \nabla\rho(x), b(x)\rangle_x = - \langle \nabla \log B_N(x), \nabla\rho(x)\rangle_x \tag{3.20}$$

i.e.

$$\langle \nabla\rho(x), b(x) + \nabla \log B_N(x)\rangle_x = 0. \tag{3.21}$$

Thus, (3.6) becomes:

$$\begin{aligned}\rho_s^\lambda = \rho(x) &+ \int_0^s d\rho(x_r^\lambda dB_r) + \int_0^s \langle \nabla\rho(x_r^\lambda), E_r^\lambda(x_r^\lambda)\rangle dr \\ &+ \frac{1}{2}\int_0^s \nabla\rho(x_r^\lambda)dr \text{ a.s. on } \{\zeta^\lambda > t\} \cap \{\zeta_N^\lambda > t\}\end{aligned} \tag{3.22}$$

$$\text{where } E_r^\lambda(x) = - \sum_{j=k+1}^{n} \frac{x_j(x)(\frac{\partial}{\partial x_j})_x}{\lambda+t-r} - \frac{1}{2}\nabla \log \theta_N(x). \tag{3.23}$$

Recall that $\rho^2(x) = \sum_{j=k+1}^{n} x_j^2(x)$; thus

$$2\rho(x)\nabla\rho(x) = 2\sum_{j=k+1}^{n} x_j(x)(\frac{\partial}{\partial x_j})_x \tag{3.24}$$

or

$$\nabla\rho(x) = \sum_{j=k+1}^{n} \frac{x_j(x)(\frac{\partial}{\partial x_j})_x}{\rho(x)} \tag{3.25}$$

Consequently,

$$\langle \nabla\rho(x), E_r(x)\rangle = - \left\langle \sum_{j=k+1}^{n} \frac{x_j(x)(\frac{\partial}{\partial x_j})_x}{\rho(x)}, \sum_{j=k+1}^{n} \frac{x_j(x)(\frac{\partial}{\partial x_j})_x}{\lambda+t-r} + \frac{1}{2}\nabla \log\theta_N(x)\right\rangle_x \tag{3.26}$$

$$= - \frac{1}{(\lambda+t-r)\rho(x)} \sum_{j=k+1}^{n} x_j^2(x) - \frac{1}{2} \langle \nabla\rho(x), \nabla \log \theta_N(x) \rangle_x \qquad (3.27)$$

$$= - \frac{\rho(x)}{\lambda+t-r} - \frac{1}{2} \frac{\partial}{\partial\rho} \log \theta_N(x). \qquad (3.28)$$

By using the Laplacian in polar Fermi coordinates,

$$\Delta\rho(x) = \frac{n-k-1}{\rho(x)} + \frac{\partial}{\partial\rho} \log \theta_N(x). \qquad (3.29)$$

Now, $\beta_s^\lambda = \int_0^s d\rho(x_r^\lambda dB_r)$ is a 1-dimensional Brownian Motion and so (3.22) becomes:

$$\rho_s^\lambda = \rho(x) + \beta_s^\lambda + \frac{n-k-1}{2} \int_0^s \frac{dr}{\rho(x_r^\lambda)} - \int_0^s \frac{\rho(x_r^\lambda)}{\lambda+t-r} dr \qquad (3.30)$$

a.s. on $\{\zeta^\lambda > t\} \cap \{\zeta_N^\lambda > t\}$.

Since $\rho(x) = |v|$ where $x = \exp_\nu(y,v)$, we also have:

$$|v_s^\lambda| = |v| + \tilde{\beta}_s^\lambda + \frac{n-k-1}{2} \int_0^s \frac{dr}{|v_r|} - \int_0^s \frac{|v_r^\lambda|}{\lambda+t-r} \qquad (3.31)$$

a.s. on $\{\zeta^\lambda > t\} \cap \{\zeta_N^\lambda > t\}$

where ζ_N^λ is now the hitting time of the origin of R^{n-k} by (v_s^λ). The S.D.E. in (3.31) is that of the Euclidean Bessel Bridge on R^{n-k} from $|v|$ to 0 in time $t + \lambda$. Hence if we choose $v \neq 0$ i.e. if we start the process away from N (which we do), then since $n-k \geq 2$, we have:

$$v_s^\lambda \neq 0 \quad \forall s \in [0,t] \quad \text{a.s.}$$

i.e.

$$\rho_s^\lambda = \rho(x_s^\lambda) = |v_s^\lambda| \neq 0 \quad \forall s \in [0,t] \text{ a.s.} \tag{3.32}$$

and so $P_x(\zeta_N^\lambda > t) = 1$ (3.33)

(3.33) says that (if we start the process away from N) the process does not hit N before and up to the time t (in fact before time t+λ). Also since $|v_s^\lambda|$ has the same S.D.E. as the Euclidean Bessel Bridge on R^{n-k}, then $(v_s^\lambda)0 \leq s \leq t$ does not explode by ([14], Chapter VI, Theorem 5). Similarly, since by (3.30), the radial behaviour of $(x_s^\lambda)0 \leq s \leq t$ is the same as that of $(v_s^\lambda)0 \leq s \leq t$, we conclude that $(x_s^\lambda)0 \leq s \leq t$ is non-explosive. Hence we have:

$$P_x(\zeta^\lambda > t) = 1 \tag{3.34}$$

and so (3.30) becomes:

$$\rho_s^\lambda = \rho(x) + \beta_s^\lambda + \frac{n-k-1}{2}\int_0^s \frac{d\tau}{\rho_r^\lambda} - \int_0^s \frac{\rho_r^\lambda}{\lambda-t-r}dr \text{ a.s.} \tag{3.35}$$

Since $(\rho_s^\lambda)0 \leq s \leq t$ and the Euclidean Bessel Bridge on R^{n-k} have the same S.D.E., they have the same distribution by ([14], Chapter VI, Theorem 6B).

We shall now proceed to prove the co-dimension 1 case. We shall work in a (global) exponential chart $(\exp_\nu^{-1}, M)$ as before. By going to the universal cover of M if necessary we can suppose that M has the structure of a trivial line bundle with a point $x = (y,\beta)$ where $y \in N$ and $\beta \in R$. Hence we have:

$$\rho(x) = d(x,N) = |\beta| .$$

Then for a C^2-function $f:M \to \mathbb{R}$ depending only on β, we have:

$$\Delta f = \frac{\partial^2 f}{\partial \beta^2} + \frac{1}{\theta_N}\frac{\partial \theta_N}{\partial \beta} \cdot \frac{\partial f}{\partial \beta} . \tag{3.36}$$

<u>Proof</u>

Set $r = |\beta|$. Then we have:

$$f(r) = \begin{cases} f(\beta) & \text{for } \beta \geq 0 \\ \\ f(-\beta) & \text{for } \beta \leq 0. \end{cases}$$

We know by Lemma (1.3) that if f depends on r alone, then,

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{\theta_N}\frac{\partial \theta_N}{\partial r} \cdot \frac{\partial f}{\partial r} . \tag{3.37}$$

Thus, for $\beta \geq 0$ (and hence $r = \beta$),

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{\theta_N}\frac{\partial \theta_N}{\partial r} \cdot \frac{\partial f}{\partial r} = \frac{\partial^2 f}{\partial \beta^2} + \frac{1}{\theta_N}\frac{\partial \theta_N}{\partial \beta} \cdot \frac{\partial f}{\partial \beta} \tag{3.38}$$

and for $\beta \leq 0$ (and hence $r = -\beta$),

$$\frac{\partial f}{\partial \beta} = - \frac{\partial f}{\partial r} ; \frac{\partial \theta_N}{\partial \beta} = - \frac{\partial \theta_N}{\partial r}; \frac{\partial^2 f}{\partial \beta^2} = \frac{\partial^2 f}{\partial r^2} .$$

Consequently,

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{\theta_N}\frac{\partial \theta_N}{\partial r} \cdot \frac{\partial f}{\partial r} = \frac{\partial^2 f}{\partial \beta^2} + \frac{1}{\theta_N}(- \frac{\partial \theta_N}{\partial \beta})(- \frac{\partial f}{\partial \beta})$$

$$= \frac{\partial^2 f}{\partial \beta^2} + \frac{1}{\theta_N}\frac{\partial \theta_N}{\partial \beta} \cdot \frac{\partial f}{\partial \beta} . \tag{3.39}$$

Consequently by (3.38) and (3.39),

$$\Delta f = \frac{\partial^2 f}{\partial \beta^2} + \frac{1}{\theta_N}\,\frac{\partial \theta_N}{\partial \beta}\cdot\frac{\partial f}{\partial \beta} \quad \forall \beta \in \mathbb{R}. \tag{3.40}$$

Set

$$Y_s^\lambda(y,\beta) = -\frac{\beta^2}{2(\lambda+t-s)} + \log C_N(y,\beta)$$

where C_N is as in (3.2). Let $x_s^\lambda = (y_s^\lambda, \beta_s^\lambda)$ be the solution of the S.D.E. with generator

$$L = \frac{1}{2}\Delta + b + A_s^\lambda$$

where

$$A_s^\lambda(y,\beta) = \nabla Y_s^\lambda(y,\beta)$$

$$= -\frac{\beta}{\lambda+t-s} + \nabla \log C_N(y,\beta).$$

(3.2(b)) <u>Lemma</u>

The process $(\beta_s^\lambda)_{0 \le s \le t}$ has the same distribution as the 1-dimensional Brownian Bridge from β to 0 in time $\lambda + t$.

<u>Proof.</u>

Let us denote by p the projection $p : N \times \mathbb{R} \to \mathbb{R}$ defined by $p(y,\beta) = \beta$. Then we know that p is C^∞ on $N \times \mathbb{R}$ and so we can apply Itô's formula on p relative to the process $x_s^\lambda = (y_s^\lambda, \beta_s^\lambda)$ with generator

$$L = \frac{1}{2}\Delta + b + A_s^\lambda:$$

$$p(y_s^\lambda,\beta_s^\lambda) = p(y_o^\lambda,\beta_o^\lambda) + \int_0^s dp(u_r^\lambda dB_r) + \int_0^s \langle \nabla p(y_r^\lambda,\beta_r^\lambda),$$

$$b(y_r^\lambda,\beta_r^\lambda) + A_r^\lambda(y_r^\lambda,\beta_r^\lambda)\rangle dr + \frac{1}{2}\int_0^s \Delta p(y_r^\lambda,\beta_r^\lambda)dr \text{ a.s.} \tag{3.41}$$

Now, $\tilde{B}_s^\lambda = \int_0^s dp(u_s^\lambda dB_r)$ is a 1-dimensional Brownian Motion. On the other hand,

$$\langle \nabla p(y,\beta), A_r^\lambda(y,\beta)\rangle = \langle \nabla p(y,\beta), -\frac{\beta}{\lambda+t-r} + \nabla \log C_N(y,\beta)\rangle \tag{3.42}$$

$$= \langle \nabla p(y,\beta), -\frac{\beta}{\lambda+t-r}\rangle - \frac{1}{2}\langle \nabla p(y,\beta), \nabla \log \theta_N(y,\beta)\rangle \tag{3.43}$$

$$+ \langle \nabla p(y,\beta), b(y,\beta) + \nabla \log B_N(y,\beta)\rangle$$

$$= -\frac{\beta}{\lambda+t-r} - \frac{1}{2}\frac{\partial}{\partial\beta}\log \theta_N(y,\beta) + \langle \nabla p(y,\beta), b(y,\beta) + \nabla \log B_N(y,\beta)\rangle. \tag{3.44}$$

Since p depends on β alone, we have by Lemma (1.3),

$$\frac{1}{2}\Delta p(y,\beta) = \frac{1}{2}\frac{1}{\theta_N(y,\beta)} \cdot \frac{\partial\theta_N}{\partial\beta}(y,\beta) = \frac{1}{2}\frac{\partial}{\partial\beta}\log \theta_N(y,\beta). \tag{3.45}$$

We must next show that:

$$\langle \nabla p, b + \nabla \log B_N\rangle = 0.$$

Recall that $\rho(x) = d(x,N) = |p(y,\beta)|$.
Thus by (3.21),

$$\langle \nabla |p(y,\beta)|, b(y,\beta) + \nabla \log B_N(y,\beta)\rangle = 0 \tag{3.46}$$

Since $|p(y,\beta)| = \begin{cases} p(y,\beta) & \text{for } \beta > 0 \\ -p(y,\beta) & \text{for } \beta < 0 \end{cases}$

we see that,

$$\langle \nabla p(y,\beta),\ b(y,\beta) + \nabla \log B_N(y,\beta)\rangle = 0 \tag{3.47}$$

for all $\beta \neq 0$ and hence for all $\beta \in \mathbb{R}$ by left and right continuity at $\beta = 0$ of the function:

$$\beta \to \langle \nabla p(y,\beta),\ b(y,\beta) + \nabla \log B_N(y,\beta)\rangle.$$

Consequently,

$$\langle \nabla p,\ b + \nabla \log B_N\rangle = 0. \tag{3.48}$$

Note also that by a direct computation, we have

$$\nabla \log B_N(y,0) = -\, b(y,0) \tag{3.49}$$

and so we have:

$$\langle \nabla p(y,0),\ b(y,0) + \nabla \log B_N(y,0)\rangle = 0 \tag{3.50}$$

and so (3.48) is still valid.

Finally by (3.44), (3.45) and (3.48), we have:

$$\beta_s^\lambda = \beta + \tilde{B}_s^\lambda - \int_0^s \frac{\beta^\lambda r}{\lambda+t-r}\, dr \quad \text{a.s.} \tag{3.51}$$

(3.51) above is just the S.D.E. for a 1-dimensional Brownian Bridge from β to 0 in time $\lambda + t$ and hence $(\beta_s^\lambda)_{0 \le s \le t}$ has the same distribution as the 1-dimensional Brownian Bridge. Consequently, $\rho_s^\lambda = \rho(x_s^\lambda) = d(x_s^\lambda, N)$ $= |\beta_s^\lambda|$ (where $x_s^\lambda = \exp_\nu(y_s^\lambda, \beta_s^\lambda)$) has the same distribution as the 1-dimensional Bessel Bridge from $|\beta|$ to 0 in time $t+\lambda$.

The next lemma is due to K.D. Elworthy and is a generalization of Lemma 1B of [18].

(3.3) Lemma

If the submanifold N is compact, then the submanifold Bridge Process $(x_s^\lambda)_{0 \le s \le t}$ converges uniformly in probability on [0,t] to the process $(x_s)_{0 \le s \le t}$ where

$$x_s = \exp_\nu(y_s^0, v_s^0) \text{ for } 0 \le s < t$$

$$x_t = y_t^0.$$

Proof.

We first consider the following Sub-Lemma:

Sub-Lemma (3.3)

Consider processes (z_s^λ), (z_s); $0 \le s \le t$ in a complete metric space B such that:

(i) The processes are sample continuous (a.s.)

(ii) $\forall\ \delta > 0$, $z_s^\lambda \to z_s$ as $\lambda \downarrow 0$ uniformly in probability on $[0,t-\delta]$

(iii) $d(z_s^\lambda, z_t) \to d(z_s, z_t)$ as $\lambda \downarrow 0$ uniformly in law on $[0,t]$.

Then $z_s^\lambda \to z_s$ as $\quad$ 0 uniformly on $[0,t]$ in probability.

Proof

Suppose $0 < \varepsilon < 1$, $0 < \delta < 2$, $\varepsilon_1 = \min(\frac{\varepsilon}{3}, \frac{\varepsilon\delta}{18})$.

Since almost sure continuity of $(z_s)_{0 \le s \le t}$ at t implies continuity of $(z_s)_{0 \le s \le t}$ at t in probability, there exists $\delta_1 = \delta_1(\varepsilon_1) > 0$ such that:

$$P_x\Big(\sup_{t-\delta_1 \le s \le t} d(z_s, z_t) > \varepsilon_1\Big) \le \frac{\varepsilon}{3} \tag{3.52}$$

By (ii), $\sup_{0 \le s \le t-\delta_1} d(z_s^\lambda, z_s) \to 0$ as $\lambda \downarrow 0$ in probability. Thus, there exists $\lambda_1 > 0$ such that for $0 < \lambda < \lambda_1$, we have:

$$P_x\Big(\sup_{0 \le s \le t-\delta_1} d(z_s^\lambda, z_s) > \delta\Big) \le \frac{\varepsilon}{3} \tag{3.53}$$

Next, set $X^\lambda = \sup_{t-\delta_1 \le s \le t} (d(z_s^\lambda, z_t) \wedge 1)$

$$X = \sup_{t-\delta_1 \le s \le t} (d(z_s, z_t) \wedge 1)$$

Then X^λ and X are random variables on $(\Omega, \underline{F}, P_x)$ with values in $[0,1]$. By (iii), $X^\lambda \to X$ as $\lambda \downarrow 0$ in law, i.e. $P_x^{X^\lambda}$ converges weakly to P_x^X as $\lambda \downarrow 0$. In particular, $E_x(X^\lambda)$ converges to $E(X)$. Hence, there exists $\delta_2 = \delta_2(\varepsilon_1) > 0$ such that for $0 < \lambda < \delta_2$, we have:

$$E_x(X^\lambda) \le E(X) + \varepsilon_1 \tag{3.54}$$

Set $\Omega(\varepsilon_1) = \{\sup_{t-\delta_1 \le s \le t} d(z_s, z_t) \le \varepsilon_1\}$.

$$\text{Then } E_x(X) = E_x(\chi_{\Omega(\varepsilon_1)} X) + E_x(\chi_{\Omega^c(\varepsilon_1)} X) \tag{3.55}$$

$$\le \varepsilon_1 + P_x(\Omega^c(\varepsilon_1)) \le \varepsilon_1 + \varepsilon_1 = 2\varepsilon_1 \tag{3.56}$$

and so by (3.54) and (3.56), we have:

$$E_x(X^\lambda) \le 3\,\varepsilon_1 \text{ for } 0 < \lambda < \lambda_2. \tag{3.57}$$

Next set $\Omega^\lambda(\delta) = \{X^\lambda > \frac{\delta}{2}\}$ for $0 < \lambda < \delta_2$.

Then for $0 < \lambda < \delta_2$, we have:

$$3\varepsilon_1 \ge E_x(X^\lambda) \ge E_x(\chi_{\Omega^\lambda(\delta)} X^\lambda) \ge \frac{\delta}{2} P_x(\Omega^\lambda(\delta)) \tag{3.58}$$

i.e.

$$P_x(\sup_{t-\delta_1 \le s \le t} (d(z_s^\lambda, z_t) \wedge 1) > \frac{\delta}{2}) \le \frac{6\varepsilon_1}{\delta} \le \frac{\varepsilon}{3} \tag{3.59}$$

for $0 < \lambda < \delta_2$ and hence, we have:

$$P_x(X^\lambda > \frac{\delta}{2}) \le \frac{\varepsilon}{3} \tag{3.60}$$

Now,

$$P_x(\sup_{0\le s\le t} d(z_s^\lambda,z_s) > \delta) \le$$
$$P_x(\sup_{0\le s\le t-\delta_1} (z_s^\lambda,z_s) > \delta) + P_x(\sup_{t-\delta_1\le s\le t} d(z_s^\lambda,z_s) > \delta) \tag{3.61}$$

By the triangular inequality,

$$\sup_{t-\delta_1\le s\le t} d(z_s^\lambda,z_s) \le \sup_{t-\delta_1\le s\le t} d(z_s^\lambda,z_t) + \sup_{t-\delta_1\le s\le t} d(z_s,z_t) \tag{3.62}$$

$$\le 2 \max (\sup_{t-\delta_1\le s\le t} d(z_s^\lambda,z_t), \sup_{t-\delta_1\le s\le t} d(z_s,z_t)). \tag{3.63}$$

The last inequality is due to the fact that for any real-valued functions f and g, we have:

$$2 \max (f,g) \ge f + g. \tag{3.64}$$

It follows that for $0 < \lambda < \lambda_0 = \min(\lambda_1,\lambda_2)$

$$P_x(\sup_{0\le s\le t} d(z_s^\lambda,z_s) > \delta) \le P_x(\sup_{0\le s\le t-\delta_1} d(z_s^\lambda,z_s) > \delta)$$
$$+ P_x(\sup_{t-\delta_1\le s\le t} d(z_s^\lambda,z_t) > \tfrac{\delta}{2}) + P_x(\sup_{t-\delta_1\le s\le t} d(z_s,z_t) > \tfrac{\delta}{2}) \tag{3.65}$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \tag{3.66}$$

by (3.53), (3.60) and (3.52).

Consider $x_s^\lambda = \exp_\nu(y_s^\lambda, v_s^\lambda)$ for $0 \le s \le t$ and $\lambda > 0$ where $(y_s^\lambda, v_s^\lambda)$ is a process in the normal bundle E_ν. If E_ν is not trivializable, then we can (by the Nash embedding Theorem) embed M isometrically into R^m for some m.

Let $TM|_N$ be the restriction of TM to N, then we have:

$$TM|_N \cong E_\nu \oplus TN \tag{3.67}$$

and so $E_\nu \subset TM_{|N} \subset N \times R^m$. (3.68)

By the exponential chart representation, we can simply set $x_s^\lambda = (y_s^\lambda, v_s^\lambda)$ for $0 \le s \le t$.

Next consider the processes $(v_s^\lambda)_{0 \le s \le t}$ and $(v_s)_{0 \le s \le t}$ where

$$v_s = v_s^0 \text{ for } 0 \le s \le t$$

$$v_t = 0 \quad \text{a.s.}$$

Then $(v_s^\lambda)_{0 \le s \le t}$ and $(v_s)_{0 \le s \le t}$ are sample continuous (a.s.) on [0,t] and so (i) is satisfied. Next $v_s^\lambda \to v_s$ as $\lambda \downarrow 0$ uniformly on $[0,t-\delta]$ in probability by ([14], §4 of Chapter VIII and Theorem 8C of Chapter VII) and so (ii) is satisfied. Lastly by the proofs of Lemmas (3.2(a)) and (3.2(b)), $|v_s^\lambda|$ is equal in distribution to the radial component of the Bessel Bridge in $R^{n-k} \subset R^m$, and hence, $|v_s^\lambda| \to |v_s|$ as $\lambda \downarrow 0$ uniformly on [0,t] in law and so (iii) is satisfied. We thus conclude by the Sub-Lemma above that $v_s^\lambda \to v_s$ as $\lambda \downarrow 0$ uniformly on [0,t] in probability.

Next consider the process $(y_s^\lambda)_{0 \le s \le t}$. Let σ_j^α be the square root of $g^{\alpha j}$ i.e.

$$g^{\alpha j} = \sum_{k=1}^{n} \sigma_k^\alpha \, \sigma_j{}^k; \; \alpha = 1,\ldots,n$$

where $(g^{\alpha j})$ are the components of the metric tensor in Fermi coordinates. Then the S.D.E. for $(y_s^\lambda)_{0 \le s \le t}$ is given by;

$$d(y_s^\lambda)_\alpha = \sum_{j=1}^{n} \sigma_j^\alpha(y_s^\lambda, v_s^\lambda) \quad dB_s^j + (b^\alpha(y^\lambda, v_s^\lambda)$$

$$- \frac{1}{2} \sum_{m,\ell=1}^{n} g^{m\ell}(y_s^\lambda, v_s^\lambda) \, \Gamma_{m\ell}^\alpha(y_s^\lambda, v_s^\lambda) + \nabla^\alpha \log C_N \, (y_s^\lambda, v_s^\lambda))ds$$

for $\alpha = 1,\ldots,k$.

We can embed N in $\mathbf{R}^\ell$ for some ℓ and since N is compact, we can extend the above coefficients smoothly to all $\mathbf{R}^\ell$. Thus we can take $N = \mathbf{R}^\ell$ and write down the stochastic Integral Equations as in (5.7) of Theorem (5.2), Chapter 5 of [19]. Thus we obtain the (uniform) limit on [0,t] in probability of $(y_s^\lambda)_{0 \le s \le t}$ to be $(y_s^0)_{0 \le s \le t}$ as in that theorem.

We finally conclude that $x_s^\lambda = (y_s^\lambda, v_s^\lambda)$ converges uniformly in probability on [0,t] to x_s as required.

§4. AN INTEGRAL FORMULA

The main result of this Chapter (the Integral Formula) will be proved in this section. This will be obtained via two preliminary theorems: One way to obtain the solution to the Cauchy Problem for the Heat Equation in M_0:

$$\frac{\partial f_t^\lambda}{\partial t} = Lf_t^\lambda$$

$$f_t^\lambda \;= 0 \text{ on the boundary of } M_o \tag{4.1}$$

$$f_0^\lambda \;= C(\lambda)f \exp\{-\frac{d(-,N)^2}{2\lambda}\}$$

(where f is continuous and of compact support in M_o) is via the integral:

$$f_t^\lambda(x) = \int_{M_o} f_o^\lambda(z)p_t^{M_o}(x,z)dz \tag{4.2}$$

where $p_t^{M_o}(-,)$ is the Heat Kernel of M_o relative to the differential operator $L = \frac{1}{2}\Delta + b + V$.

(4.1) <u>Theorem</u>

$$\lim_{\lambda\downarrow 0} f_t^\lambda(x) = \int_N f(y)p_t^{M_o}(x,y)dy$$

<u>Proof</u>.

$$f_t^\lambda(x) = \int_{M_o} f_o^\lambda(z)p_t^{M_o}(x,z)dz \tag{4.3}$$

$$= \int_{E_\nu^o} f_o^\lambda(\exp_\nu(y,v))p_t^{M_o}(x,\exp_\nu(y,v))\theta_N(y,v)dydv. \tag{4.4}$$

Since $E_\nu^o \subset N \times R^{n-k}$ and f_o^λ has compact support in $M_o = \exp_\nu^{-1}(E_\nu^o)$, we have, after setting $v = \sqrt{\lambda}\,\omega$ and choosing $C(\lambda) = (2\pi\lambda)^{-(n-k)/2}$:

$$f_t^\lambda(x) = (2\pi)^{\frac{n-k}{2}}\int_{N\times\mathbb{R}^{n-k}} f(\exp_y(\sqrt{\lambda}\omega))\exp\{-\frac{\|\omega\|^2}{2}\}p_t^{M_o}(x,\exp_y(\sqrt{\lambda}\omega))\theta_N(y,\sqrt{\lambda}\omega)dyd\omega$$

Then, taking limits as $\lambda \downarrow 0$, we have:

$$\lim_{0} f_t^\lambda(x) = (2\pi)^{-\frac{n-k}{2}}\int_{N\times\mathbb{R}^{n-k}} f(\exp_y 0)\exp\{-\frac{\|\omega\|^2}{2}\}p_t^{M_o}(x,\exp_y 0)\theta_N(y,0)dyd\omega \tag{4.7}$$

Now, $\exp_y 0 = y$ and by ([21], Proposition (2.2)), $\theta_N(y,0) = 1$ and so:

$$\lim_{\lambda\downarrow 0} f_t^\lambda(x) = (2\pi)^{-\frac{n-k}{2}}\int_{\mathbb{R}^{n-k}} \exp\{-\frac{\|\omega\|^2}{2}\}d\omega \int_N f(y)p_t^{M_o}(x,y)dy \tag{4.8}$$

$$= \int f(y)p_t^{M_o}(x,y)dy \tag{4.9}$$

since

$$\int_{\mathbb{R}^{n-k}} \exp\{-\frac{\|\omega\|^2}{2}\}\,d\omega = (2\pi)^{\frac{n-k}{2}} \tag{4.10}$$

Another way to obtain the solution of the Cauchy Problem for the Heat Equation in M_o (given in (4.1)) is as follows:

$$f_t^\lambda(x) = \mathbb{E}_x(\chi_{\zeta^\lambda>t}\, f_o^\lambda(x_t^\lambda)M_t^\lambda \exp\{\int_0^t V(x_s^\lambda)ds\}) \tag{4.11}$$

by the Girsanov-Cameron-Martin-Feynman-Kac formula where $(x_s^\lambda)_{0\le s\le t}$ is the Semi-Classical Brownian Riemannian Bridge from x to the submanifold N in time $t+\lambda$ and ζ^λ is its first exit time from M_o.

M_t^λ is the exponential local martingale given by

$$M_t^\lambda = \exp\{-\int_0^t \langle A_s^\lambda(x_s^\lambda), u_s^\lambda dB_s\rangle_{x_s^\lambda} - \frac{1}{2}\int_0^t \|A_s^\lambda(x_s^\lambda)\|^2 ds\} \tag{4.12}$$

where $(u_s^\lambda)_{0 \le s \le t}$ is the horizontal lift of the process $(x_s^\lambda)_{0 \le s \le t}$ on the frame bundle $O(M)$ to M. $(B_s)_{0 \le s < +\infty}$ is (of course) the n-dimensional Euclidean Brownian Motion.

(4.2) Theorem

$$\lim_{\lambda \downarrow 0} f_t^\lambda(x) = (2\pi t)^{-\frac{n-k}{2}} \exp\{-\frac{d(x,N)^2}{2t}\}.\, C_N(x)$$

$$\times\, E_x(\chi_{\zeta>t}\, C_N^{-1}(x_t^t) f(x_t^t) \exp\{\int_0^t \frac{LC_N}{C_N}(x_s^t)ds\})$$

where $(x_s^t)_{0 \le s \le t}$ is the Semi-Classical Brownian Riemannian Bridge from x to N in time t i.e. the process on M_0 with generator

$\frac{1}{2}\Delta + b + A_s$ where

$$A_s(x) = A_s^0(x) = -\sum_{j=k+1}^{n} \frac{x_j(x)(\frac{\partial}{\partial x_j})_x}{t-s} + \nabla \log C_N(x) \tag{4.13}$$

and ζ is the first exit time from M_0 which we assume to have compact closure.

Proof.

Much of the proof is based on the computations on the exponential local martingale M_t^λ. Recall that Y_s^λ is defined (in Fermi coordinates) by

$$Y_s^\lambda(x) = -\frac{\rho^2(x)}{2(\lambda+t-s)} + \log C_N(x), \tag{4.14}$$

then $\nabla Y_s = A_s^\lambda$. (4.15)

Itô's formula on Y_s^λ gives:

$$Y_t^\lambda(x_t^\lambda) = Y_o^\lambda(x) + \int_0^t \frac{\partial Y_s^\lambda}{\partial s}(x_s^\lambda)ds + \int_0^t \langle \nabla Y_s^\lambda(x_s^\lambda), \overset{\lambda}{s}\, dB_s\rangle \tag{4.16}$$

$$+ \int_0^t \langle \nabla Y_s^\lambda(x_s^\lambda), b(x_s^\lambda) + A_s^\lambda(x_s^\lambda)\rangle ds + \frac{1}{2}\int_0^t \Delta Y_s^\lambda(x_s^\lambda)ds.$$

Now, using (4.16) above, we can substitute for $\int_0^t \langle A_s^\lambda(x_s^\lambda), \mu_s^\lambda\, dB_s\rangle ds$ in the expression for M_t^λ to obtain:

$$M_t^\lambda = \exp\{Y_o^\lambda(x) - Y_t^\lambda(x_t^\lambda) + \int_0^t \frac{\partial Y_s}{\partial s}(x_s^\lambda)ds + \frac{1}{2}\int_0^t \|A_s^\lambda(x_s)\|^2 ds$$

$$+ \int_0^t \langle A_s^\lambda(x_s^\lambda), b(x_s^\lambda)\rangle ds + \frac{1}{2}\int_0^t \Delta Y_s^\lambda(x_s^\lambda))ds\} \tag{4.17}$$

Now, $\frac{\partial Y_s^\lambda}{\partial s}(x) = -\frac{\rho^2(x)}{2(\lambda+t-s)}$ and since,

$$A_s^\lambda(x) = \nabla Y_s^\lambda(x) = -\frac{\rho(x)\nabla\rho(x)}{\lambda+t-s} + \nabla \log C_N(x),$$

we have:

$$\frac{1}{2}\|A_s^\lambda(x)\|^2 = \frac{1}{2}\langle A_s^\lambda(x), A_s^\lambda(x)\rangle$$

$$= \frac{\rho^2(x)}{2(\lambda+t-s)}\|\nabla\rho(x)\|^2 - \langle\nabla\rho(x), \nabla \log C_N(x)\rangle \frac{\rho(x)}{\lambda+t-s} + \frac{1}{2}\|\nabla \log C_N(x)\|^2 \tag{4.18}$$

$$\rho^2(x) = \sum_{j=k+1}^{n} x_j^2(x) . \tag{4.19}$$

Hence $\nabla\rho^2(x) = 2\rho(x)\nabla\rho(x) = 2\,\Sigma\, x_j(x)\left(\frac{\partial}{\partial x_j}\right)_x$ (4.20)

and so, $\nabla\rho(x) = \sum_{j=k+1}^{n} \frac{x_j(x)}{\rho(x)}\left(\frac{\partial}{\partial x_j}\right)_x$ (4.21)

Consequently, $\|\nabla\rho(x)\|^2 = \frac{1}{\rho^2(x)} \sum_{j=k+1}^{n} x_j^2(x)$ (4.22)

$$= 1 \tag{4.23}$$

Thus

$$\frac{\partial Y_s^\lambda}{\partial s}(x) + \frac{1}{2}\|A_s^\lambda(x)\|^2 = -\frac{\rho(x)}{\lambda+t-s}\frac{\partial}{\partial\rho}\log C_N(x) + \frac{1}{2}\|\nabla \log C_N(x)\|^2 \tag{4.24}$$

Now, using the Laplacian in polar Fermi coordinates,

$$\Delta Y_s^\lambda(x) = -\frac{1}{2(\lambda+t-s)}[2 + (\frac{n-k-1}{\rho(x)} + \frac{\partial}{\partial\rho}\log \theta_N(x))2\rho(x)] + \Delta \log C_N(x). \tag{4.25}$$

Hence, $\frac{1}{2}\Delta Y_s^\lambda(x) = -\frac{1}{2(\lambda+t-s)}[(n-k) + \rho(x)\frac{\partial}{\partial\rho}\log \theta_N(x)] + \frac{1}{2}\Delta \log C_N(x)$ (4.26)

Nex $\langle\nabla Y_s^\lambda(x), b(x)\rangle = \langle -\frac{\rho(x)\nabla\rho(x)}{\lambda+t-s} + \nabla \log C_N(x), b(x)\rangle$

$$= -\frac{\rho(x)}{\lambda+t-s}\langle\nabla\rho(x), b(x)\rangle + \langle \nabla \log C_N(x), b(x)\rangle \tag{4.27}$$

Now, $-\langle\nabla\rho(x), b(x)\rangle = -\langle\nabla\rho(x), b(x) + \nabla \log B_N(x)\rangle + \langle\nabla\rho(x), \nabla \log B_N(x)\rangle$.

Since $\langle\nabla\rho(x),\, b(x) + \nabla \log B_N(x)\rangle = 0$ by (3.21), we have:

$$\langle\nabla Y_s^\lambda(x),\, b(x)\rangle = \frac{\rho(x)}{\lambda+t-s}\langle\nabla\rho(x),\, \nabla \log B_N(x)\rangle \tag{4.28}$$

$$+ \langle\nabla \log C_N(x),\, b(x)\rangle$$

$$= \frac{\rho(x)}{\lambda+t-s}\frac{\partial}{\partial\rho}\log B_N(x) + \langle \nabla \log C_N(x),\, b(x)\rangle \tag{4.29}$$

Therefore by (4.26) and (4.29), we have:

$$\langle\nabla Y_s^\lambda(x),\, b(x)\rangle + \frac{1}{2}\Delta Y_s^\lambda(x)$$

$$= -\frac{n-k}{2(\lambda+t-s)} + \frac{\rho(x)}{\lambda+t-s}\frac{\partial}{\partial\rho}\log\theta_N^{-\frac{1}{2}} + \frac{1}{2}\Delta\log C_N(x) \tag{4.30}$$

$$+ \frac{\rho(x)}{\lambda+t-s}\frac{\partial}{\partial\rho}\log B_N(x) + \langle\nabla \log C_N(x),\, b(x)\rangle$$

$$= -\frac{n-k}{2(\lambda+t-s)} + \frac{\rho(x)}{\lambda+t-s}\frac{\partial}{\partial\rho}\log(B_N(x)\theta_N^{-\frac{1}{2}}) + (\frac{1}{2}\Delta + b)\log C_N(x) \tag{4.31}$$

Consequently by (4.24) and (4.31),

$$M_t^\lambda = \exp\{-\frac{\rho^2(x)}{2(\lambda+t)} + \frac{\rho^2(x_t^\lambda)}{2\lambda} + \log C_N(x) - \log C_N(x_t^\lambda) \tag{4.32}$$

$$+ \int_0^t (\frac{1}{2}\|\nabla \log C_N(x_s^\lambda) + \frac{1}{2}\Delta\log C_N(x_s^\lambda) + b(\log C_N)(x_s^\lambda) - \frac{n-k}{2(\lambda+t-s)})ds\}$$

$$= (\frac{\lambda}{\lambda+t})^{\frac{n-k}{2}}\exp\{-\frac{d(x,N)^2}{2(\lambda+t)} + \frac{d(x_t^\lambda,N)^2}{2\lambda}\}\frac{C_N(x)}{C_N(x_t^\lambda)} \tag{4.33}$$

$$\times \exp\{\int_0^t (\tfrac{1}{2}\|\nabla \log C_N(x_s)\|^2 + \tfrac{1}{2}\Delta \log C_N^\lambda(x_s) + b(\log C_N^\lambda)(x_s))ds\}.$$

Now,

$$\frac{1}{2}\|\nabla \log C_N\|^2 + \frac{1}{2}\Delta \log C_N$$

$$= \frac{1}{2}\|\nabla \log B_N\|^2 + \frac{1}{2}\Delta \log B_N - \frac{1}{2}\langle \nabla \log B_N, \nabla \log \theta_N\rangle$$

$$+ \frac{1}{8}\|\nabla \log \theta_N\|^2 - \frac{1}{4}\Delta \log \theta_N \tag{4.34}$$

We know that:

$$\frac{1}{8}\|\nabla \log \theta_N\|^2 - \frac{1}{4}\Delta \log \theta_N = \frac{1}{2}\theta_N^{\frac{1}{2}}\Delta\theta_N^{-\frac{1}{2}} \tag{4.35}$$

and by setting $B_N = D_N^{-\frac{1}{2}}$, we have:

$$\frac{1}{2}\|\nabla \log B_N\|^2 + \frac{1}{2}\Delta \log B_N = \frac{1}{8}\|\nabla \log D_N\|^2 - \frac{1}{4}\Delta \log D_N$$

$$= \frac{1}{2} D_N^{\frac{1}{2}}\Delta D_N^{-\frac{1}{2}} \tag{4.36}$$

$$= \frac{1}{2} B_N^{-1}\Delta B_N \tag{4.37}$$

Consequently, (4.34) becomes by (4.35) and (4.37):

$$\frac{1}{2}\|\nabla \log C_N\|^2 + \frac{1}{2}\Delta \log C_N = \frac{1}{2}\theta_N^{\frac{1}{2}}\Delta\theta_N^{-\frac{1}{2}} + \frac{1}{2}B_N^{-1}\Delta B_N - \frac{1}{2}\langle \frac{\nabla B_N}{B_N}, \frac{\nabla\theta_N}{\theta_N}\rangle$$

$$= \frac{1}{2}\frac{\Delta(B_N\theta_N^{-\frac{1}{2}})}{B_N\,\theta_N^{-\frac{1}{2}}} = \frac{1}{2}\frac{\Delta C_N}{C_N}. \tag{4.38}$$

Since $b(\log C_N) = \langle b, \frac{\nabla C_N}{C_N}\rangle$, the expression in (4.33) becomes:

$$M_t^\lambda = \left(\frac{\lambda}{\lambda+t}\right)^{\frac{n-k}{2}} \frac{C_N(x)}{C_N(x_t^\lambda)} \exp\left\{\frac{d(x,N)^2}{2(\lambda+t)} + \frac{d(x_t^\lambda,N)}{2\lambda}\right\}$$

$$\times \exp\left\{\int_0^t \left(\frac{1}{2}\frac{\Delta C_N(x_s^\lambda)}{C_N(x_s^\lambda)} + \langle b(x_s^\lambda), \frac{\nabla C_N(x_s^\lambda)}{C_N(x_s^\lambda)}\rangle\right)ds\right\} \tag{4.39}$$

Recalling the expression for f_0^λ and choosing $C(\lambda) = (2\pi\lambda)^{-\frac{n-k}{2}}$ as before,

$$f_t^\lambda(x) = (2\pi(\lambda+t))^{-\frac{n-k}{2}} C_N(x)\exp\left\{-\frac{d(x,N)^2}{2(\lambda+t)}\right\}$$

$$\times E_x(\chi_{\zeta^\lambda>t} C_N^{-1}(x_t^\lambda)f(x_t^\lambda) \exp\left\{\int_0^t \frac{LC_N}{C_N}(x_s^\lambda)ds\right\}) \tag{4.40}$$

Then taking limits as $\lambda \downarrow 0$, we have:

$$\lim_{\lambda\downarrow 0} f_t^\lambda(x) = (2\pi t)^{-\frac{n-k}{2}} C_N(x)\exp\left\{-\frac{d(x,N)^2}{2t}\right\} \tag{4.41}$$

$$\times \lim_{\lambda\downarrow 0} E_x(\chi_{\zeta^\lambda>t}\, C_N^{-1}(x_t^\lambda)f(x_t^\lambda)\exp\left\{\int_0^t \frac{LC_N}{C_N}(x_s^\lambda)ds\right\}).$$

The limit on the L.H.S. of (4.41) exists by Theorem (4.1) and so that of the R.H.S. exists too. Then assuming that M_0 has compact closure, we can compute inequalities as in Theorems 1.5(a) and 1.5(b) of Chapter I to obtain the equality of the Theorem.

(4.3) Theorem (An Integral Formula)

Let M_o be of compact closure and f of support in M_o, then

$$\int_N f(y)p_t^{M_o}(x,y)dy = (2\pi t)^{-\frac{n-k}{2}} C_N(x)\exp\{-\frac{\rho^2(x)}{2t}\}$$

$$\times E_x(\chi_{\zeta>t}C_N^{-1}(x_t^t)f(x_t^t)\exp\{\int_0^t \frac{LC_N}{C_N}(x_s^t)ds\}).$$

Proof.

Obvious by Theorem (4.1) and Theorem (4.2).

(4.4) Remark

Recall (see Lemma (3.3)) that $x_t^t = y_t^o$ is the limit in probability of $x_t^\lambda \in N$ as $\lambda \downarrow 0$. Consequently there exists a subsequence $(x_t^{\lambda_k})_{k\geq 1}$ such that $x_t^{\lambda_k} \to y_t^o$ a.s. and hence

$$x_t^t = y_t^o \in N \quad \text{a.s.} \tag{4.42}$$

Hence,

$$B_N(x_t^t) = B_N(y_t^o) = 1 \quad \text{a.s.} \tag{4.43}$$

from the definition of B_N.

Similarly,

$$\theta_N(x_t^t) = \theta_N(y_t^o,0) = 1 \quad \text{a.s.} \tag{4.44}$$

by ([21], Prop. (2.2)) and so,

$$C_N^{-1}(x_t^t) = 1 \text{ a.s.} \tag{4.45}$$

We can thus re-write the result of Theorem (4.3) as:

(4.3 (bis)) Theorem

$$\int_N f(y)p_t^{M_o}(x,y)dy = (2\pi t)^{-\frac{n-k}{2}} C_N(x) \exp\{-\frac{d(x,N)^2}{2t}\}$$

$$\times\ E_x(\chi_{\zeta>t}\ f(y_t^o)\exp\{\int_0^t \frac{LC_N}{C_N}(x_s^t)ds\}).$$

(4.5) Remark

All the examples of the submanifold N that we will give will consist of complete submanifolds and hence we can safely assume that N is complete and hence closed as a subspace of the metric space M.

(4.6) Corollary

If N is any (complete) submanifold (not necessarily compact) and f is of compact support in M_o, then we still have:

$$\int_N f(y)p_t^{M_o}(x,y)dy = (2\pi t)^{-\frac{n-k}{2}} C_N(x)\exp\{-\frac{d(x,N)^2}{2t}\}$$

$$\times\ E_x(\chi_{\zeta>t}\ f(y_t^o)\exp\{\int_0^t \frac{LC_N}{C_N}(x_s^t)ds\}).$$

Thus the Expectation on the R.H.S. is finite since the integral on the L.H.S. is.

Proof.

Let M_o^j be an increasing sequence of compact domains of M_o exhausting M_o and let ζ_j be the first exit time of the Bridge Process $(x^t(s))_{0 \le s \le t \wedge \zeta}$

from M_o^j. Then $\zeta_j \uparrow \zeta$ and we have by setting $N_j = N \cap M_o^j$ (hence N_j is a compact manifold with boundary but the boundary causes no problem)

$$\int_{N_j} f(y)p_t^{M_o^j}(x,y)dy = (2\pi t)^{-\frac{n-k}{2}} C_N(x) \exp\{-\frac{d(x,N_j)^2}{2t}\}$$
$$\times E_x(\chi_{\zeta_j>t}\, f(y_t^o)\exp\{\int_0^t \frac{LC_N}{C_N}(x_s^t)ds\}). \tag{4.46}$$

Now, letting j tend to $+\infty$ in (4.46) above, we obtain the result since $N_j \uparrow N$ as $j \uparrow \infty$ and

$$C_{N_j}(x) = C_N(x) \text{ for } x \in M^j.$$

§5. SOME EXAMPLES

In the examples of this section we will assume (to simplify computations) that $b \equiv 0$. What is essential in the computation of the above formula is the computation of the expression $\frac{1}{2}\theta_N^{\frac{1}{2}}\Delta\theta_N^{-\frac{1}{2}}$. In general, the change of volume factor θ_N under the exponential map of the normal bundle $\exp_\nu$ is given by (Lemma (6.1) of [20]):

$$\frac{\partial}{\partial\rho}(\log \theta_N(y,\rho\xi)) = -(\frac{n-k-1}{\rho} + \text{trace } S(\rho)) \tag{5.1}$$

where $\xi \in S^{n-k-1}(1)$ and $S(\rho)$ is the Second Fundamental Form of the hypersurface:

$$\partial M_o(\rho) = \{m \in M_o : d(m,N) = \rho\} \tag{5.2}$$
$$0 \le \rho < c(y,\xi)$$

$$\text{trace } S(\rho) = \sum_{i=1;\ i\neq k+1}^{n} K_i(\rho) \tag{5.3}$$

$$= (n-1)H(\rho) \tag{5.4}$$

where $K_1(\rho), K_2(\rho), \ldots, K_k(\rho), K_{k+2}(\rho), \ldots, K_n(\rho)$

are the principal curvatures of the hypersurface $\partial M_o(\rho)$ and $H(\rho)$ its mean curvature.

When the manifold is complete and its sectional curvature $K^M = K$ is constant, then we get a more precise expression for θ_N.

First we have the following Lemma of Alfred Gray in ([20], Lemma (6.2)).

(5.1) Lemma

Define the functions $\phi(\lambda,\rho,\xi)$ for $\xi \in S^{n-k-1}(1)$ and $0 \leq \rho \leq \rho_o < c(y,\xi)$:

$$\phi(\lambda,\rho,\xi) = \begin{cases} \left(\dfrac{\sin\rho\sqrt{\lambda}}{\rho\sqrt{\lambda}}\right)^{n-k-1} \det\left(\cos\rho\sqrt{\lambda}\, I_{k,k} - \dfrac{\sin\rho\sqrt{\lambda}}{\sqrt{\lambda}}\, \Pi_{k,k}(\xi)\right) & \text{for } \lambda > 0 \\[2ex] \det\left(I_{k,k} - \rho\,\Pi_{k,k}(\xi)\right) & \text{for } \lambda = 0 \\[2ex] \left(\dfrac{\sinh\rho\sqrt{|\lambda|}}{\rho\sqrt{|\lambda|}}\right)^{n-k-1} \det\left(\cosh\rho\sqrt{|\lambda|}\, I_{k,k} - \dfrac{\sinh\rho\sqrt{|\lambda|}}{\sqrt{|\lambda|}}\, \Pi_{k,k}(\xi)\right) & \text{for } \lambda < 0 \end{cases}$$

where $I_{k,k}$ is the unit $k \times k$ matrix and $\Pi_{k,k}(\xi)$ is the $k \times k$ matrix associated to the Second Fundamental Form $\Pi(\xi)$ of the k-dimensional submanifold N of M; $0 \leq k \leq n-1$.

Then (since M is complete) we have:

(i) If $K^M = K \geq \lambda$, then $\theta_N(y,\rho\xi) \leq \phi(\lambda,\rho,\xi)$

(ii) If $K^M = K \leq \lambda$, then $\theta_N(y,\rho\xi) \geq \phi(\lambda,\rho,\xi)$.

(5.2) Theorem

Let M be complete and of constant sectional curvature $K^M = K$; then

(i) For $K > 0$, we have:

$$\theta_N(y,\rho\xi) = \left(\frac{\sin \rho\sqrt{K}}{\rho\sqrt{K}}\right)^{n-k-1} \det(\cos \rho\sqrt{K}\ I_{k,k} - \frac{\sin\ \rho\sqrt{K}}{\sqrt{K}}\ \Pi_{k,k}(\xi))$$

for $0 \leq \rho \leq \rho_o < \frac{\Pi}{\sqrt{K}}$.

(ii) For $K = 0$, we have:

$$\theta_N(y,\rho\xi) = \det(I_{k,k} - \rho\Pi_{k,k}(\xi))$$

for $0 \leq \rho \leq \rho_o < +\infty$.

(iii) For $K < 0$, we have:

$$\theta_N(y,\rho\xi) = \left(\frac{\sinh \rho\sqrt{|K|}}{\rho\sqrt{|K|}}\right)^{n-k-1} \det(\cosh \rho\sqrt{|K|}.\ I_{k,k} - \frac{\sinh \rho\ \sqrt{|K|}}{\sqrt{K}}\ \Pi_{k,k}(\xi))$$

for $0 < \rho \leq \rho_o < +.\infty$.

Proof.

We take $K = \lambda$ = constant and use the above Lemma.

(5.3) Example

Take $M = R^n$ and N any orientable hypersurface whose principal curvatures are $K_1(\rho), K_2(\rho), \ldots, K_{n-1}(\rho)$, then,

$$\theta_N(y,\rho\xi) = \prod_{i=1}^{n-1} (1-\rho K_i(\rho)) \qquad (5.5)$$

by Theorem (5.2(ii)).

In particular, for $N = S^{n-1}$, then

$$\theta_N(y,\rho\xi) = (1+\rho)^{n-1} \qquad (5.6)$$

since $K_i(\rho) = -1$.

The next examples will mostly be concerned with totally geodesic submanifolds.

(5.4) Definition

The submanifold N is totally geodesic in M if every geodesic of N is also a geodesic in M. This is characterized by the fact that the Second Fundamental Form vanishes identically.

From now henceforth we let $S^n = S^n(1)$.

(5.5) Example

Let $M = S^n$ and $N = S^k$ for $1 \le k \le n-1$. All the S^k are "equitorial" and are totally geodesic in S^n. Thus the Second Fundamental Forms vanish identically and by Theorem (5.2) and the fact that the sectional curvature $K = 1$, we have:

$$\theta_N(y,\rho\xi) = (\frac{\sin\rho}{\rho})^{n-k-1}\,(\cos\rho)^k \tag{5.7}$$

for $-\frac{\Pi}{2} < \rho < \frac{\Pi}{2}$.

Note that it is possible to compute for "non-equitorial" spheres.

We next compute the expression $\frac{1}{2}\,\theta_N^{\frac{1}{2}}\Delta\theta_N^{-\frac{1}{2}}$

$$\Delta\theta_N^{-\frac{1}{2}} = \Delta[(\frac{\sin\rho}{\rho})^{-\frac{n-k-1}{2}}(\cos\rho)^{-\frac{k}{2}}] \tag{5.8}$$

$$= (\cos\rho)^{-\frac{k}{2}}\,\Delta(\frac{\sin\rho}{\rho})^{-\frac{n-k-1}{2}} + (\frac{\sin\rho}{\rho})^{-\frac{n-k-1}{2}}\,\Delta(\cos\rho)^{-\frac{k}{2}}$$

$$+ 2\,\langle\nabla(\frac{\sin\rho}{\rho})^{-\frac{n-k-1}{2}},\ \nabla(\cos\rho)^{-\frac{k}{2}}\rangle \tag{5.9}$$

Therefore, we have:

$$\frac{1}{2}\theta_N^{\frac{1}{2}}\Delta\theta_N^{-\frac{1}{2}} = \frac{1}{2}(\frac{\sin\rho}{\rho})^{\frac{n-k-1}{2}}\,\Delta(\frac{\sin\rho}{\rho})^{-\frac{n-k-1}{2}} + \frac{1}{2}\,(\cos\rho)^{\frac{k}{2}}\,\Delta(\cos\rho)^{-\frac{k}{2}} \tag{5.10}$$

$$+ \langle\frac{\nabla\,(\frac{\sin\rho}{\rho})^{-\frac{n-k-1}{2}}}{(\frac{\sin\rho}{\rho})^{-\frac{n-k-1}{2}}},\ \frac{\nabla\,(\cos\rho)^{-\frac{k}{2}}}{(\cos\rho)^{-\frac{k}{2}}}\rangle$$

$$= A + B + C$$

where $A = \frac{1}{2}(\frac{\sin\rho}{\rho})^{\frac{n-k-1}{2}}\,\Delta(\frac{\sin\rho}{\rho})^{-\frac{n-k-1}{2}}$ (5.11)

$$= \frac{(n-k-1)^2}{8} + \frac{(n-k-1)(n-k-3)}{8}(\frac{1}{\rho^2} - \frac{1}{\sin^2\rho}) \tag{5.12}$$

by using the Laplacian in Fermi coordinates (polar version).

By using the Laplacian in Fermi Coordinates again,

$$\Delta(\cos\rho)^{-\frac{k}{2}} = \frac{\partial^2}{\partial\rho^2}(\cos\rho)^{-\frac{k}{2}} + \left(\frac{n-k-1}{\rho} + \frac{\partial}{\partial\rho}\log\theta_N\right)\frac{\partial}{\partial\rho}(\cos\rho)^{-\frac{k}{2}} \tag{5.13}$$

$$= \frac{k}{2}\left[\frac{k+2}{2}(\cos\rho)^{-\frac{k+4}{2}}\sin^2\rho + (\cos\rho)^{-\frac{k}{2}}\right]$$

$$+ \left[\left(\frac{n-k-1}{\rho}\right) + (n-k-1)\left(\frac{\cos\rho}{\sin\rho} - \frac{1}{\rho}\right) - k\,\frac{\sin\rho}{\cos\rho}\right]\frac{k}{2}(\cos\rho)^{-\frac{k+2}{2}}\sin\rho \tag{5.14}$$

$$= \frac{k}{2}\left[\frac{k+2}{2}(\cos\rho)^{-\frac{k+4}{2}}\sin^2\rho + (\cos\rho)^{-\frac{k}{2}}\right]$$

$$+ \frac{k}{2}(n-k-1)(\cos\rho)^{-\frac{k}{2}} - \frac{k}{2}\cdot k\,(\cos\rho)^{-\frac{k+4}{2}}\sin^2\rho \tag{5.15}$$

Hence, $B = \frac{1}{2}(\cos\rho)^{\frac{k}{2}}\Delta(\cos\rho)^{-\frac{k}{2}}$

$$= \frac{k}{4}\left[\frac{k+2}{2}(\cos\rho)^{-2}(\sin\rho)^2 + 1\right] + \frac{k}{4}(n-k-1) - \frac{k}{4}\cdot k(\cos\rho)^{-2}(\sin\rho)^2$$

$$= -\frac{k}{4}\left(\frac{k-2}{2}\right)\tan^2\rho + \frac{k}{4}(n-k) \tag{5.16}$$

$$C = \left\langle \nabla\log\left(\frac{\sin\rho}{\rho}\right)^{-\frac{n-k-1}{2}},\ \nabla\log(\cos\rho)^{-\frac{k}{2}}\right\rangle \tag{5.17}$$

$$= \frac{k}{2}\left(\frac{n-k-1}{2}\right)\left\langle \nabla\log\left(\frac{\sin\rho}{\rho}\right),\ \nabla\log(\cos\rho)\right\rangle \tag{5.18}$$

$$= \frac{k}{2}\left(\frac{n-k-1}{2}\right)\left\langle \frac{\cos\rho}{\sin\rho}\nabla\rho - \frac{\nabla\rho}{\rho},\ -\frac{\sin\rho}{\cos\rho}\nabla\rho\right\rangle \tag{5.19}$$

$$= -\frac{k}{2}\left(\frac{n-k-1}{2}\right)\|\nabla\rho\|^2 + \frac{k}{2}\left(\frac{n-k-1}{2}\right)\frac{\sin\rho}{\rho\cos\rho}\|\nabla\rho\|^2 \tag{5.20}$$

$$= -\frac{k}{2}\left(\frac{n-k-1}{2}\right) + \frac{k}{2}\left(\frac{n-k-1}{2}\right)\frac{\sin\rho}{\rho\cos\rho} \tag{5.21}$$

since $\|\nabla\rho\| = 1$.

Thus by (5.12), (5.16) and (5.21),

$$\frac{1}{2}\,\theta_N^{\frac{1}{2}}\Delta\theta_N^{-\frac{1}{2}} = A + B + C$$

$$= \frac{(n-k-1)^2}{8} + \frac{(n-k-1)(n-k-3)}{8}\left(\frac{1}{\rho^2} - \frac{1}{\sin^2\rho}\right) - \frac{k}{4}\left(\frac{k-2}{2}\right)\tan^2\rho \qquad (5.22)$$

$$+ \frac{k}{4}(n-k) - \frac{k}{2}\left(\frac{n-k-1}{2}\right) + \frac{k}{2}\left(\frac{n-k-1}{2}\right)\frac{\tan\rho}{\rho}$$

$$= \frac{(n-k-1)^2}{8} + \frac{(n-k-1)(n-k-3)}{8}\left(\frac{1}{\rho^2} - \frac{1}{\sin^2\rho}\right) \qquad (5.23)$$

$$+ \frac{k}{4} - \frac{k}{4}\left(\frac{k-2}{2}\right)\tan^2\rho + \frac{k}{4}(n-k-1)\frac{\tan\rho}{\rho}.$$

Thus by the Integral Formula (Theorem (4.3)), we have for $1 \le k \le n-1$,

$$\int_{S^k} p_t^{S_0^n}(x,y)dy = (2\pi t)^{-\frac{n-k}{2}} \exp\left\{-\frac{\rho^2(x)}{2t}\right\}\left(\frac{\sin\rho(x)}{\rho(x)}\right)^{-\frac{n-k-1}{2}}(\cos\rho(x))^{-\frac{k}{2}} \qquad (5.24)$$

$$\times\ E_x\Big(\chi_{\zeta_0>t}\exp\Big\{\int_0^t \Big(\frac{k}{4} + \frac{(n-k-1)^2}{8} + \frac{(n-k-1)(n-k-3)}{8}\Big(\frac{1}{\rho^2(x_s)} - \frac{1}{\sin^2\rho(x_s)}\Big)$$

$$- \frac{k(k-2)}{8}\tan^2\rho(x_s) + \frac{k(n-k-1)}{4}\frac{\tan\rho(x_s)}{\rho(x_s)} + V(x_s)\Big)ds\Big\}\Big)$$

where S_0^n is a tubular neighbourhood of S^k in S^n of radius $\rho_0 \in (0,\frac{\pi}{2})$ about S^k.

(5.6) Remark

In the above integral, the expressions on the R.H.S. involve only the radial part $\rho(x_s)$ of the submanifold Bridge Process when $V \equiv 0$. We know from Lemmas (3.2)(a) and (3.2)(b) that $\rho(x_s)$ is isonomous to the radial part of the (n-k)-dimensional Euclidean Bridge. Hence in the above example, as in all the examples that will follow, things reduce to Euclidean Bridges (when we suppose $V \equiv 0$): $\zeta = \zeta(t)$ is the first exit time of Euclidean Bridge from a neighbourhood.

(5.7) Proposition

We have:

$$\int_{S^k} p_t^{S^n}(x,y)dy = (2\pi t)^{-\frac{n-k}{2}} \exp\{-\frac{\rho^2(x)}{2t}\} \left(\frac{\sin\rho(x)}{\rho(x)}\right)^{-\frac{n-k-1}{2}} (\cos\rho(x))^{-\frac{k}{2}}$$

$$\times E_x(\chi_{\zeta>t} \exp\{\int_0^t (\frac{k}{4} + \frac{(n-k-1)^2}{8} + \frac{(n-k-1)(n-k-3)}{8}(\frac{1}{\rho^2(x_s)} - \frac{1}{\sin^2\rho(x_s)})$$

$$- \frac{k(k-2)}{8}\tan^2\rho(x_s) + \frac{k(n-k-1)}{4}\frac{\tan\rho(x_s)}{\rho(x_s)} + V(x_s))ds\})$$

where $\zeta = \zeta(t)$ is the first hitting time of $\{p,\bar{p}\}$ by the submanifold Bridge Process $(x_s)0 \le s \le t \wedge \zeta$. p is the North pole and $\bar{p}$ the South pole and $x \notin \{p,\bar{p}\}$. When $V \equiv 0$, then ζ is the same as the first exit time of the (n-k)-dimensional Euclidean Brownian Bridge from the ball $D(0,\frac{\pi}{2})$.

Proof

Let $(S_o^n)_j$ be an increasing sequence of tubular neighbourhoods of S^k in S^n and let ζ_j be the corresponding sequence of first exit times from $(S_o^n)_j$. Then,

$$(\mathbb{S}^n_0)_j \uparrow \mathbb{S}^n \smallsetminus \{p,\bar{p}\} \text{ is } j \uparrow \infty \text{ and } \zeta_j \uparrow \zeta \text{ as } j \uparrow \infty .$$

Now, the Integral Formula in (5.24) is true for each $j \geq 1$, and so taking limits as $j \uparrow \infty$, we have

$$\int_{\mathbb{S}^k} p_t^{\mathbb{S}^n \smallsetminus \{p,\bar{p}\}}(x,y)dy = \text{R.H.S. of Proposition} \tag{5.25}$$

$$\text{But } p_t^{\mathbb{S}^n \smallsetminus \{p,\bar{p}\}} = p_t^{\mathbb{S}} \text{ on } (\mathbb{S}^n \smallsetminus \{p,\bar{p}\}) \times (\mathbb{S}^n \smallsetminus \{p,\bar{p}\}) \tag{5.26}$$

since $\{p,\bar{p}\}$ has capacity zero in $\mathbb{S}^n$ for $n \geq 2$.

The Proposition is thus proved.

In particular, we have:

$$\int_{\mathbb{S}^2} p_t^{S^3}(x,y)dy = (2\pi t)^{-\frac{1}{2}} \exp\{-\frac{\rho^2(x)}{2t}\} \frac{e^{\frac{t}{2}}}{\cos \rho(x)} P_x(\zeta > t)$$

where $\zeta = \zeta(t)$ is the first hitting time of $\{p,\bar{p}\}$ in $\mathbb{S}^3$ by the submanifold Bridge Process $(x_s)0 \leq s \leq t \wedge \zeta$. As pointed out in Remark (5.6), we have:

$$P_x(\zeta > t) = P_\beta(\zeta_0 > t)$$

where $\zeta_0 = \zeta_0(t)$ is the first exit time from $]-\frac{\pi}{2}, \frac{\pi}{2}[$ of the 1-dimensional Brownian Bridge (B_s^t) from β to 0 in time t where $x = \exp_\nu(y,\beta)$.

We know by ([24]) that for $0 < \varepsilon \leq \frac{\pi}{2}$,

$$P_\beta(\sup_{0\leq s\leq t} |B_s^t| < \varepsilon) = 1-2 \sum_{n=1}^{+\infty} (-1)^{n+1}\exp\{-\frac{2n^2\varepsilon^2}{t}\} \cosh(\frac{2n\varepsilon\beta}{t}).$$

Recall that $\rho(x) = |\beta|$. Also, it is trivially clear that

$\cosh(\frac{2n\varepsilon\beta}{t}) = \cosh(\frac{2n\varepsilon|\beta|}{t})$ and so $\cosh(\frac{2n\varepsilon\beta}{t}) = \cosh(\frac{2n\varepsilon\rho(x)}{t})$.
Consequently,

$$\int_{S^2} p_t^{S^3}(x,y)dy = (2\pi t)^{-\frac{1}{2}} \exp\{-\frac{\rho^2(x)}{2t}\} \frac{e^{\frac{t}{2}}}{\cos\rho(x)}$$

$$\times (1-2\sum_{n=1}^{\infty} (-1)^{n+1} \exp\{-\frac{n^2\pi^2}{2t}\} \cosh(\frac{n\pi\rho(x)}{t})).$$

(5.8) Example

Let $M = H^n$ = the Hyperbolic n-space and $N = H^k$ where $1 \le k \le n-1$. Then each H^k is totally geodesic in H^n. Then by Theorem (5.2) and the fact that $K = -1$,

$$\theta_N(y,\rho\xi) = (\frac{\sinh\rho}{\rho})^{n-k-1} (\cosh\rho)^k \tag{5.27}$$

Let us compute $\frac{1}{2}\theta_N^{\frac{1}{2}}\Delta\theta_N^{-\frac{1}{2}}$:

$$\Delta\theta_N^{-\frac{1}{2}} = \Delta[(\frac{\sinh\rho}{\rho})^{-\frac{n-k-1}{2}} (\cosh\rho)^{-\frac{k}{2}}] \tag{5.28}$$

$$= (\cosh\rho)^{-\frac{k}{2}} \Delta(\frac{\sinh\rho}{\rho})^{-\frac{n-k-1}{2}} + (\frac{\sinh\rho}{\rho})^{-\frac{n-k-1}{2}} \Delta(\cosh\rho)^{-\frac{k}{2}}$$

$$+ 2 < \nabla(\frac{\sinh\rho}{\rho})^{-\frac{n-k-1}{2}}, \nabla(\cosh\rho)^{-\frac{k}{2}}> \tag{5.29}$$

Hence, $\frac{1}{2}\theta_N^{\frac{1}{2}}\Delta\theta_N^{-\frac{1}{2}} = \frac{1}{2}(\frac{\sinh\rho}{\rho})^{\frac{n-k-1}{2}} \Delta(\frac{\sinh\rho}{\rho})^{-\frac{n-k-1}{2}}$

$$+ \frac{1}{2}(\cosh\rho)^{\frac{k}{2}} \Delta(\cosh\rho)^{-\frac{k}{2}} + <\nabla \log(\frac{\sinh\rho}{\rho})^{-\frac{n-k-1}{2}}, \nabla\log(\cosh\rho)^{-\frac{k}{2}}> \tag{5.30}$$

$= A + B + C.$

$$A = \frac{1}{2}\left(\frac{\sinh\rho}{\rho}\right)^{\frac{n-k-1}{2}} \Delta\left(\frac{\sinh\rho}{\rho}\right)^{-\frac{n-k-1}{2}} \tag{5.31}$$

$$= -\frac{(n-k-1)^2}{8} + \frac{(n-k-1)(n-k-3)}{8}\left(\frac{1}{\rho^2} - \frac{1}{\sinh^2\rho}\right)$$

$B = \frac{1}{2}(\cosh\rho)^{\frac{k}{2}} \Delta(\cosh\rho)^{-\frac{k}{2}}$. By using the Laplacian in polar Fermi coordinates,

$$\Delta(\cosh\rho)^{-\frac{k}{2}} = \frac{k}{2}\left[\frac{k+2}{2}(\cosh\rho)^{-\frac{k+4}{2}}\sinh^2\rho - (\cosh\rho)^{-\frac{k}{2}}\right] \tag{5.32}$$

$$+ \left[\frac{n-k-1}{\rho} + (n-k-1)\left(\frac{\cosh}{\rho} - \frac{1}{\rho}\right) + \frac{k\sinh\rho}{\cosh\rho}\right]\left[-\frac{k}{2}(\cosh\rho)^{-\frac{k+2}{2}}\sinh\rho\right]$$

$$= \frac{k}{2}\left(\frac{k+2}{2}\right)(\cosh\rho)^{-\frac{k+4}{2}}\sinh^2\rho - \frac{k}{2}(\cosh\rho)^{-\frac{k}{2}} - \frac{k}{2}(n-k-1)(\cosh\rho)^{-\frac{k}{2}}$$

$$- \frac{k}{2}k(\cosh\rho)^{-\frac{k+4}{2}}\sinh^2\rho. \tag{5.33}$$

Hence, $B = \frac{1}{2}(\cosh\rho)^{\frac{k}{2}} \Delta(\cosh\rho)^{-\frac{k}{2}}$

$$= \frac{1}{2}\left[\frac{k}{2}\left(\frac{k+2}{2}\right)\cosh^{-2}\rho\ \sinh^2\rho - \frac{k}{2} - \frac{k}{2}(n-k-1) - \frac{k}{2}\cdot k\cosh^{-2}\rho\sinh^2\rho\right] \tag{5.34}$$

$$= \frac{1}{2}\left[\frac{k}{2}\left(\frac{k+2}{2}\right)\tanh^2\rho - \frac{k}{2}(n-k) - \frac{k}{2}\cdot k\tanh^2\rho\right] \tag{5.35}$$

$$= -\frac{k}{4}\left(\frac{k-2}{2}\right)\tanh^2\rho - \frac{k}{4}(n-k) \tag{5.36}$$

$$C = \langle \nabla \log \left(\frac{\sinh \rho}{\rho}\right)^{-\frac{n-k-1}{2}}, \nabla \log (\cosh \rho)^{-\frac{k}{2}} \rangle \tag{5.37}$$

$$= \frac{k}{2}\left(\frac{n-k-1}{2}\right) \langle \nabla \log\left(\frac{\sinh \rho}{\rho}\right), \nabla \log(\cosh \rho)\rangle \tag{5.38}$$

$$= \frac{k}{2}\left(\frac{n-k-1}{2}\right) \langle \frac{\cosh \rho}{\sinh \rho} \nabla\rho - \frac{\nabla\rho}{\rho}, \frac{\sinh \rho}{\cosh \rho} \nabla\rho \rangle \tag{5.39}$$

$$= \frac{k}{2}\left(\frac{n-k-1}{2}\right) \|\nabla\rho\|^2 - \frac{k}{2}\left(\frac{n-k-1}{2}\right)\frac{\sinh \rho}{\rho\cosh\rho} \|\nabla\rho\|^2 . \tag{5.40}$$

By the Generalized Gauss Lemma ([20], Lemma 3.1), $\nabla\rho = \vec{n}$ where $\vec{n}$ is a unit (normal) vector field: $\vec{n}(x) \in (T_yN)^{\perp}$ where y is the point at which the unit speed geodesic from x meets N orthogonally:

$$\vec{n}(x) = \sum_{i=k+1}^{n} \frac{x_i}{\rho(x)} \left(\frac{\partial}{\partial x_i}\right)_x$$

and so $\|\vec{n}(x)\| = 1$. Hence, we have:

$$C = \frac{k}{4}(n-k-1) - \frac{k}{4}(n-k-1) \frac{\tanh \rho}{\rho} \tag{5.41}$$

Finally, $\frac{1}{2}\, \theta_N^{\frac{1}{2}}\Delta\theta_N^{-\frac{1}{2}} = A + B + C$

$$\begin{aligned} &= -\frac{(n-k-1)^2}{8} + \frac{(n-k-1)(n-k-3)}{8}\left(\frac{1}{\rho^2} - \frac{1}{\sinh^2\rho}\right) \\ &\quad - \frac{k}{8}(k-2)\tanh^2\rho - \frac{k}{4}(n-k) + \frac{k}{4}(n-k-1) - \frac{k}{4}(n-k-1)\frac{\tanh \rho}{\rho} \end{aligned} \tag{5.42}$$

$$\begin{aligned} &= -\frac{k}{4} - \frac{(n-k-1)^2}{8} + \frac{(n-k-1)(n-k-3)}{8}\left(\frac{1}{\rho^2} - \frac{1}{\sinh^2\rho}\right) \\ &\quad - \frac{k}{8}(k-2)\tanh^2\rho - \frac{k}{4}(n-k-1)\frac{\tanh \rho}{\rho}. \end{aligned} \tag{5.43}$$

(5.9) Proposition

$$\int_{H^k} p_t^{H^n}(x,y)dy = (2\pi t)^{-\frac{n-k}{2}} \exp\{-\frac{\rho^2(x)}{2t}\} \left(\frac{\sinh\rho(x)}{\rho(x)}\right)^{-\frac{n-k-1}{2}} (\cosh\rho(x))^{-\frac{k}{2}}$$

$$\times E_x(\exp\{\int_0^t (-\frac{k}{4} - \frac{(n-k-1)^2}{8} + \frac{(n-k-1)(n-k-3)}{8}(\frac{1}{\rho^2(x_s)} - \frac{1}{\sinh^2\rho(x_s)})$$

$$- \frac{k}{8}(k-2)\tanh^2\rho(x_s) - \frac{k}{4}(n-k-1)\frac{\tanh\rho(x_s)}{\rho(x_s)} + V(x_s))ds\}).$$

Proof.

The result follows by the Integral Formula (Corollary (4.6)). Note that the Bridge Process is non-explosive and hence the explosion time $\zeta = +\infty$.

Note also that the expression on the R.H.S. of the above integral is in terms of the (n-k)-dimensional Euclidean Brownian Bridge when $V \equiv 0$.

In particular we deduce from Proposition (5.9) above that

$$\int_{H^2} p_t^{H^3}(x,y)dy = (2\pi t)^{-\frac{1}{2}} \exp\{-\frac{\rho^2(x)}{2t}\} \frac{e^{-\frac{t}{2}}}{\cosh\rho(x)} \tag{5.44}$$

Recall that $p_t^{H^3}(x,y) = (2\pi t)^{-\frac{3}{2}} e^{-\frac{t}{2}} \frac{r(x)}{\sinh r(x)} \exp\{-\frac{r^2(x)}{2t}\}$

where $r(x) = d(x,y)$. Thus from (5.44) we have:

$$\int_{H^2} \frac{d(x,y)}{\sinh d(x,y)} \exp\{-\frac{d(x,y)^2}{2t}\}dy = 2\pi t \exp\{-\frac{d(x,H^2)^2}{2t}\} \frac{1}{\cosh d(x,H^2)} \tag{5.45}$$

If $x \in H^2$, then clearly the R.H.S. of (5.45) above becomes $2\pi t$. In this case we can verify by an easy computation: We set $y = \exp_x v$. Then the above integral becomes:

$$\int_{\mathbb{R}^2} \frac{d(x,\exp_x v)}{\sinh d(x,\exp_x v)} \exp\{-\, d(x,\exp_x v)^2\}\theta_x(v)dv \tag{5.46}$$

$$= \int_{\mathbb{R}^2} \frac{\|v\|}{\sinh\|v\|} \cdot \exp\{-\frac{\|v\|^2}{2t}\,\frac{\sinh\|v\|}{\|v\|}\,dv \tag{5.47}$$

$$= \int_{\mathbb{R}^2} \exp\{-\frac{\|v\|^2}{2t}\}\, dv = 2\pi t \tag{5.48}$$

(5.10) Example

We take $M = \mathbb{C}P^n$ = the 2n-dimensional Complex Projective Space and let $N = \mathbb{C}P^k$ for $1 \leq k \leq n-1$. Again $\mathbb{C}P^k$ is totally geodesic in $\mathbb{C}P^n$ ([8]; Chap. 3, D, §3.24).

Now $$\theta_N(y,\rho\xi) = (\frac{\sin\rho}{\rho})^{2n-2k-1}(\cos\rho)^{2k+1} \tag{5.49}$$

by ([22]; Proof of Theorem 8).

Then computing as in Example (5.5), we have:

$$\begin{aligned}\frac{1}{2}\theta_N^{\frac{1}{2}}\,\Delta\theta_N^{-\frac{1}{2}} &= \frac{2k+1}{4} + \frac{(2n-2k-1)^2}{8} + \frac{(2n-2k-1)(2n-2k-3)}{8}(\frac{1}{\rho^2} - \frac{1}{\sin^2\rho}) \\ &- \frac{(2k+1)(2k-1)}{8}\tan^2\rho + \frac{(2k+1)(2n-2k-1)}{4}\,\frac{\tan\rho}{\rho}\end{aligned} \tag{5.50}$$

Fix a point $x_o \in \mathbb{C}P^n$. Then by ([8], Chap. 3, D, Proposition 3.35) $\mathbb{C}P^{n-1}$ is the submanifold in $\mathbb{C}P^n$ antipodal to x_o. Next consider the (largest) tubular neighbourhood

$$\mathbb{C}P^n_o = \{x \in \mathbb{C}P^n : d(x,\mathbb{C}P^k) < \frac{\pi}{2}\} \tag{5.51}$$

of $\mathbb{C}P^k$ in $\mathbb{C}P^n$. Hence the set of cut-focal points is:

$$\{x \in \mathbb{C}P^n : d(x, \mathbb{C}P^k) = \tfrac{\pi}{2}\} = \{x_o\} \tag{5.52}$$

and hence has co-dimension 2n and so Brownian Motion starting from $x \neq x_o$ never hits x_o.

Consequently by ([16]; p. 109),

$$p_t^{\mathbb{C}P_o}(x,y) = p_t^{\mathbb{C}P^n}(x,y) \text{ for all } (x,y) \notin \{x_o\} \times \{x_o\} \tag{5.53}$$

Since the Integral Formula is valid for the tubular neighbourhood $\mathbb{C}P_o$, we have:

(5.11) Proposition

$$\int_{\mathbb{C}P^k} p_t^{\mathbb{C}P^n}(x,y)dy = (2\pi t)^{-(n-k)} \exp\{-\frac{\rho^2(x)}{2t}\} \left(\frac{\sin\rho(x)}{\rho(x)}\right)^{-\frac{(2n-2k-1)}{2}} (\cos\rho(x))^{-\frac{2k+1}{2}}$$

$$\times E_x(\chi_{\zeta>t} \exp\{ \int_0^t (\frac{2k+1}{4} + \frac{(2n-2k-1)^2}{8} + \frac{(2n-2k-1)(2n-2k-3)}{8}(\frac{1}{\rho^2(x_s)} - \frac{1}{\sin^2\rho(x_s)})$$

$$\frac{(2k+1)(2k-1)}{8} \tan^2\rho(x_s) + \frac{(2k+1)(2n-2k-1)}{4} \frac{\tan\rho(x_s)}{\rho(x_s)} + V(x_s))ds\})$$

for $\rho \in (0, \frac{\pi}{2})$ where $\zeta = \zeta(t)$ is the first exit time of the (2n-2k)-dimensional Euclidean Brownian Bridge from the (Euclidean) ball $D(0, \frac{\pi}{2})$ (when $V \equiv 0$).

(5.12) Example

The case of the 4n-dimensional Quaternionic Projective Space $\mathbb{H}P^n$ is dealt with similarly. $\mathbb{H}P^k$ is totally goedesic in $\mathbb{H}P^n$ for $1 \le k \le n-1$. By ([22]; proof of Theorem 8),

$$\theta_N(y,\rho\xi) = \left(\frac{\sin \rho}{\rho}\right)^{4n-4k-1} (\cos \rho)^{4k+3} \tag{5.54}$$

Computing as in the case of the Complex Projective Space, we obtain:

$$\int_{\mathbb{H}P^k} p_t^{\mathbb{H}P^n}(x,y)dy = (2\pi t)^{-2(n-k)} \exp\{-\frac{\rho^2(x)}{2t}\}\left(\frac{\sin \rho(x)}{\rho(x)}\right)^{-\frac{(4n-4k-1)}{2}} (\cos\rho(x))^{-\frac{4k+3}{2}}$$

$$\times E_x(\chi_{\zeta>t} \exp\{ \int_0^t (\frac{4k+3}{4} + \frac{(4n-4k-1)^2}{8} + \frac{(4n-4k-1)(4n-4k-3)}{8}(\frac{1}{\rho^2(x_s)} - \frac{1}{\sin^2\rho(x_s)})$$

$$- \frac{(4k+3)(4k+1)}{8}\tan^2\rho(x_s) + \frac{(4k+3)(4n-4k-1)}{4}\, \frac{\tan \rho(x_s)}{\rho(x_s)} + v(x_s))ds\}).$$

§6. SOME MORE APPLICATIONS - RIEMANNIAN SUBMERSIONS AND HEAT KERNELS

We next come to some of the important applications of the main Theorem (the Integral Formula of the Heat Kernel). From it we will obtain the stochastic representation of the Heat Kernel for $\mathbb{C}P^n$ and $\mathbb{H}P^n$. The tool for this is the Theorem below:

(6.1) Theorem ([14]; Chap. IX, Theorem 10E)

Let $\pi: M \to B$ be a Riemannian submersion with minimal fibres in M. If $(x_t)0 \le t < \zeta(x)$ is Brownian Motion in M starting at $x \in M$, then $\pi(x_t)0 \le t < \zeta(x)$ is equal in distribution to Brownian Motion in B starting at $\pi(x) \in B$ and restricted to $[0,\zeta(x)) \times \Omega$.

Denote by p_t^M the Heat Kernel of M and by p_t^B that of B. Then a consequence of the above Theorem is that:

$$p_t^B (\pi(x),\pi(y)) = \int_{\pi^{-1}(q)} p_t^M(x,z)dz \tag{6.1}$$

where $q = \pi(y)$.

(6.2) Proposition

$$p_t^{\mathbb{C}P^n}(\pi(x),\pi(y)) = (2\pi t)^{-n} \exp\{-\frac{\rho^2(x)}{2t}\} \left(\frac{\sin\rho(x)}{\rho(x)}\right)^{-\frac{2n-1}{2}} (\cos\rho(x))^{-\frac{1}{2}}$$

$$\times\ E_x(\chi_{\zeta>t} \exp\{\int_0^t (\frac{1}{4} + \frac{(2n-1)^2}{8} + \frac{(2n-1)(2n-3)}{8}\left(\frac{1}{\rho^2(x_s)} - \frac{1}{\sin^2\rho(x_s)}\right)$$

$$+ \frac{\tan^2\rho(x_s)}{8} + \frac{(2n-1)}{4}\frac{\tan\rho(x_s)}{\rho(x_s)} + V(x_s))ds\})$$

where $\rho = d(-,S^1)$ and $(x_s)0 \le s \le t \wedge \zeta$ is the submanifold Brownian Riemannian Bridge in S^{2n+1} starting at $x \in S^{2n+1}$ and reaching the submanifold S^1 at time t.

Proof.

We apply (6.1) to the Riemannian submersion:

$$\pi: S^{2n+1} \longrightarrow \mathbb{C}P^n.$$

Thus we have:

$$p_t^{\mathbb{C}P^n}(\pi(x),\pi(y)) = \int_{S^1} p_t^{S^{2n+1}}(x,z)dz \qquad (6.2)$$

since $S^1 = \pi^{-1}(\pi(y))$. The Integral Formula (Theorem (4.3)) via Proposition (5.7) then gives the result.

We note that when $V \equiv 0$, we can take $(x_s)0 \le s \le t \wedge \zeta_0$ to be the 2n-dimensional Euclidean Brownian Bridge.

The case of the 4n-dimensional Quaternionic Projective Space $\mathbb{H}P^n$ is dealt with similarly.

(6.3) Proposition

$$p_t^{\mathbb{H}P^n}(\pi(x),\pi(y)) = (2\pi t)^{-2n} \exp\{-\frac{\rho^2(x)}{2t}\} \left(\frac{\sin\rho(x)}{\rho(x)}\right)^{-\frac{4n-1}{2}} (\cos \rho(x))^{-\frac{3}{2}}$$

$$\times\, E_x(\chi_{\zeta>t} \exp\{ \int_0^t (\frac{3}{4} + \frac{(4n-1)^2}{8} + \frac{(4n-1)(4n-3)}{8}(\frac{1}{\rho^2(x_s)} - \frac{1}{\sin^2\rho(x_s)})$$

$$- \frac{3 \tan^2 \rho(x_s)}{8} + \frac{3(4n-1)\tan \rho(x_s)}{\rho(x_s)} + V(x_s))ds\}).$$

Proof.

We use the Riemannian submersion

$$\pi : S^{4n+3} \longrightarrow \mathbb{H}P^n$$

whose(minimal) fibres are the 3-spheres S^3:

By (6.1) we have:

$$p_t^{\mathbb{H}P^n}(\pi(x),\pi(y)) = \int_{S^3} p_t^{S^{4n+3}}(x,z)dz \tag{6.3}$$

We obtain the result by the Integral formula via Proposition (5.7). When $V \equiv 0$, we can take $(x_s)0 \leq s \leq t \wedge \zeta_0$ to be 4n-dimensional Euclidean Brownian Bridge starting from $x \in R^{4n}$ and ending at 0 in time t.

CHAPTER III : EXACT AND ASYMPTOTIC EXPANSIONS OF THE INTEGRAL OF THE HEAT KERNEL OVER A SUBMANIFOLD

§0. INTRODUCTION

Set $r_t(x,N) = (2\pi t)^{-\frac{n-k}{2}}\ \theta_N^{-\frac{1}{2}}(x)\ \exp\{-\frac{\rho^2(x)}{2t}\}$.

Following the same lines as in [37], we will prove the following Lemma:

(0.1) Lemma

$$\frac{\partial r_t}{\partial t}(x,N) = \frac{1}{2}\Delta r_t(x,N) - \frac{\Delta\theta_N^{-\frac{1}{2}}}{2\theta_N^{-\frac{1}{2}}}(x) r_t(x,N).$$

Proof

It is well known that for real-valued functions f, we have:

$$\Delta \exp(f) = \{\Delta f + \|\nabla f\|^2\} \exp(f) \tag{0.1}$$

Now, $r_t(x,N) = (2\pi t)^{-\frac{n-k}{2}} \exp\{\log \theta_N^{-\frac{1}{2}}(x) - \frac{\rho^2(x)}{2t}\}$. (0.2)

Hence using (0.1) we get after some simplifications:

$$\frac{1}{2}\Delta r_t(-,N) = \frac{1}{2}\left[\frac{\Delta\theta_N^{-\frac{1}{2}}}{\theta_N^{-\frac{1}{2}}} - \frac{n-k}{t} + \frac{\rho^2}{t^2}\right] r_t(-,N). \tag{0.3}$$

On the other hand,

$$\frac{\partial r_t}{\partial t}(-,N) = \left[-\frac{n-k}{2t} + \frac{\rho^2}{2t^2}\right] r_t(-,N). \tag{0.4}$$

We thus get the result by (0.3) and (0.4).

§1. SEMI-CLASSICAL SEMIGROUPS FOR SUBMANIFOLD BRIDGE PROCESSES

Set $q_t(x,N) = B_N(x) r_t(x,N)$.

(1.1) Lemma

$$\frac{\partial q_t}{\partial t}(x,N) = Lq_t(x,N) - \frac{LC_N}{C_N}(x) q_t(x,N).$$

Proof

$$\frac{\partial q_t}{\partial t}(-,N) = B_N \frac{\partial r_t}{\partial t}(-,N) = B_N\Big[\frac{1}{2}\Delta r_t(-,N) - \frac{\Delta\theta_N^{-\frac{1}{2}}}{2\theta_N^{-\frac{1}{2}}} r_t(-,N)\Big] \tag{1.1}$$

$$\Delta q_t(-,N) = B_N \Delta r_t(-,N) + r_t(-,N)\Delta B_N + 2\langle \nabla r_t(-,N), \nabla B_N\rangle \tag{1.2}$$

Using (1.2) we substitute for $B_N \Delta r_t(-,N)$ in (1.1) to get:

$$\frac{\partial q_t}{\partial t}(-,N) = \frac{1}{2}\Delta q_t(-,N) - \frac{1}{2} r_t(-,N)\Delta B_N - \langle \nabla r_t(-N), \nabla B_N\rangle - B_N \frac{\Delta\theta_N^{-\frac{1}{2}}}{2\theta_N^{-\frac{1}{2}}} r_t(-,N) \tag{1.3}$$

$$= \frac{1}{2}\Delta q_t(-,N) - (2\pi t)^{-\frac{n-k}{2}} \exp\{-\frac{\rho^2}{2t}\}\Big[\frac{1}{2}\theta_N^{-\frac{1}{2}}\Delta B_N + \frac{1}{2} B_N \Delta\theta_N^{-\frac{1}{2}}\Big] - \langle \nabla r_t(-,N), \nabla B_N\rangle$$

$$= \frac{1}{2}\Delta q_t(-,N) - (2\pi t)^{-\frac{n-k}{2}} C_N \exp\{-\frac{\rho^2}{2t}\}\Big[\frac{1}{2}\frac{\Delta C_N}{C_N} - \Big\langle \frac{\nabla B_N}{B_N}, \frac{\nabla\theta_N^{-\frac{1}{2}}}{\theta_N^{-\frac{1}{2}}}\Big\rangle\Big] - \langle \nabla r_t(-,N), \nabla B_N\rangle$$

$$= \frac{1}{2}\Delta q_t(-,N) - \frac{1}{2}\frac{\Delta C_N}{C_N} q_t(-,N) + \langle \nabla \log B_N, \nabla \log \theta_N^{-\frac{1}{2}}\rangle q_t(-,N)$$

$$- \langle \nabla r_t(-,N), \nabla B_N\rangle \tag{1.4}$$

But

$$<\nabla r_t(-,N),\nabla B_N> = <\nabla \log r_t(-,N),\nabla \log B_N> q_t(-,N)$$

$$= <-\frac{\nabla\rho^2}{2t},\nabla \log B_N> q_t(-,N) + <\nabla \log \theta_N^{-\frac{1}{2}}, \nabla \log B_N> q_t(-,N). \tag{1.5}$$

Hence by (1.4) and (1.5) we have:

$$\frac{\partial q_t}{\partial t}(-,N) = \frac{1}{2}\Delta q_t(-,N) - \frac{1}{2}\frac{\Delta C_N}{C_N}\cdot q_t(-,N) + \frac{\rho}{t}<\nabla\rho, \nabla \log B_N>$$

$$= Lq_t(-,N) - \frac{LC_N}{C_N}\cdot q_t(-,N) - <b,\nabla \log q_t(-,N) - \nabla \log C_N> q_t(-,N)> + \frac{\rho}{t}<\nabla\rho,\nabla \log B_N>$$

$$= Lq_t(-,N) - \frac{LC_N}{C_N}\cdot q_t(-,N) + \frac{\rho}{t}<b + \nabla \log B_N,\nabla\rho>$$

since $<b, \nabla \log q_t(-,N) - \nabla \log C_N> q_t(-,N) = -\frac{\rho}{t}<b,\nabla\rho>$.

But by (3.21) of Chapter II, $<b + \nabla \log B_N,\nabla\rho> = 0$ and so:

$$\frac{\partial q_t}{\partial t}(-,N) = Lq_t(-,N) - \frac{LC_N}{C_N}\cdot q_t(-,N).$$

Define for $t \geq s > 0$ the operators $Q_N^{M_o}(t,s)$ on $C^2(M)$-functions with compact support in M_o as follows:

$$(Q_N^{M_o}(t,s)f)(x) = q_t(x,N)^{-1}\cdot P_{t-s}^{M_o}(q_s(-,N)f)(x) \tag{1.6}$$

where $(P_t^{M_o})t \geq 0$ is the Markov Semigroup of operators associated to the Heat Kernel $p_t^{M_o}(-,-)$. Thus the R.H.S. of (1.6) becomes:

$$q_t(x,N)^{-1} \int_{M_o} q_s(z,N)f(z)p_{t-s}^{M_o}(x,z)dz \tag{1.7}$$

(1.2) Lemma

$$\frac{\partial}{\partial t}(Q_N^{M_o}(t,s)f)(x) = [L-V + \nabla \log q_t(-,N) + \frac{LC_N}{C_N}](Q_N^{M_o}(t,s)f)(x)$$

Proof.

The proof follows the same lines asthose of Lemma (2.3) of [37]:

$$\frac{\partial}{\partial t}(Q_N^{M_o}(t,s)f)(x) = \frac{\partial}{\partial t} (q_t(x,N)^{-1} P_{t-s}^{M_o}(q_s(-,N)f)(x)) \tag{1.8}$$

$$= \quad - q_t(x,N)^{-2} \frac{\partial q_t}{\partial t} (x,N) P_{t-s}^{M_o}(q_s(-,N)f)(x)$$

$$+ q_t(x,N)^{-1} \frac{\partial}{\partial t} (P_{t-s}^{M_o}(q_s(-,N)f)(x)) \tag{1.9}$$

$$= -q_t(x,N)^{-2} Kq_t(x,N)\cdot P_{t-s}^{M_o}(q_s(-,N)f)(x)$$

$$+ q_t(x,N)^{-1} L(P_{t-s}^{M_o}(q_s(-,N)f)(x). \tag{1.10}$$

The first expression on the R.H.S. (1.10) is due to Lemma (1.1) above and the second expression to Theorem (1.6) of [4].

(1.10) is equal to:

$$[-\frac{Lq_t(x,N)}{q_t(x,N)} + \frac{LC_N}{C_N}(x)](Q_N^{M_o}(t,s)f)(x)$$

$$+ q_t(x,N)^{-1}[(Q_N^{M_o}(t,s)f)(x)\cdot Lq_t(x,N) + q_t(x,N)\cdot L(Q_N^{M_o}(t,s)f)(x) \tag{1.11}$$

$$+ \langle\nabla q_t(x,N),\ \nabla(Q_N^{M_o}(t,s)f)(x)\rangle - V(x)\cdot q_t(x,N)\cdot(Q_N^{M_o}(t,s)f)(x)]$$

$$= [(L-V) + \nabla \log q_t(-,N) + \frac{LC_N}{C_N}](Q_N^{M_o}(t,s)f)(x) \tag{1.12}$$

as can be seen directly.

(1.3) Proposition

For $f \in C^2(M)$ with compact support in M_o and $t \geq s > 0$, we have:

$$(Q_N^{M_o}(t,t-s)f)(x) = E_x(\chi_{\zeta>s}f(x^t(s))\exp\{\int_0^s \frac{LC_N}{C_N}(x^t(r))dr\}))$$

where $(x^t(s))0 \leq s \leq t \wedge \zeta$ is the submanifold Bridge Process defined in Chapter II.

Proof.

The proof follows similar arguments as those of Proposition (2.7) in [37].

Define the process $(\omega^t(r))\ r \geq 0$ on M_o by:

$$\omega^t(r) = x^t(r \wedge \zeta). \tag{1.13}$$

Further define the process $(y(r))0 \leq r \leq s$ by:

$$y:[0,s] \times \Omega \to [t-s,t] \times \bar{M}_0 \times R$$

$$y(r) = (t-r,\ \omega^t(r),\ v(r)) \tag{1.14}$$

where $v(r) = \exp\{\int_0^r \frac{LC_N}{C_N}(\omega^t(u))du\}$ (1.15)

i.e. $dv(r) = \dfrac{LC_N(\omega^t(r))v(r)}{C_N(\omega^t(r))} dr$. (1.16)

Finally define,

$$g_\alpha(x) = (Q_N^{M_0}(\alpha,t-s)f)(x) \text{ for } \alpha \geq t-s > 0 \tag{1.17}$$

and

$$h(\alpha,x,v) = g_\alpha(x)v \text{ for } v \geq 0. \tag{1.18}$$

Then

$$h(y(r)) = h(t-r,\omega^t(r),v(r)) = g_{t-r}(\omega^t(r))v(r) \tag{1.19}$$

and

$$dh(y(r)) = dg_{t-r}(\omega^t(r))v(r) + g_{t-r}(\omega^t(r))dv(r) \tag{1.20}$$

$$= [\tfrac{\partial g}{\partial r}t\text{-}r(\omega^t(r))dr + \langle\nabla g_{t-r}(\omega^t(r)),\ u_r dB_r\rangle_{\omega^t(\sigma)}$$

$$+ \langle b(\omega^t(r)) + \nabla \log q_{t-r}(\omega^t(r),N),\ \nabla g_{t-r}(\omega^t(r))\rangle_{\omega^t(r)} dr \tag{1.21}$$

$$+ \tfrac{1}{2}\Delta g_{t-r}(\omega^t(r))dr]v(r) + g_{t-r}(\omega^t(r))\ \frac{LC_N(\omega^t(\ \ ))v(r)}{C_N(\omega^t(r))}dr \quad \text{a.s.}$$

where $(u_r)_{0 \leq r < +\infty}$ is the horizontal lift of $(\omega^t(r))_{0 \leq r < +\infty}$

on the frame bundle $O(M_o)$ of M_o. The above is Itô's formula in differential form.

Set $r' = t-r$, then

$$\frac{\partial g_{t-r}}{\partial r}(\omega^t(r)) = -\frac{\partial g_{r'}}{\partial r'}(\omega^t(r)) \tag{1.22}$$

$$= -\frac{\partial}{\partial r'}(Q_N^{M_o}(r',t-s)f)(\omega^t(r)). \tag{1.23}$$

By Lemma (1.2), (1.23) becomes:

$$-[(L-V) + \nabla \log q_{r'}(-,N) + \frac{LC_N}{C_N}](Q_N^{M_o}(r',t-s)f)(\omega^t(r)) \tag{1.24}$$

$$= -[(L-V) + \nabla \log q_{t-r}(-,N) + \frac{LC_N}{C_N}](Q_N^{M_o}(t-r,t-s)f)(\omega^t(r)). \tag{1.25}$$

Recalling that $g_{t-r}(\omega^t(r)) = (Q_N^{M_o}(t-r,t-s)f)(\omega^t(r))$ we substitute (1.25) in (1.21) to obtain:

$$dh(y(r)) = \langle \nabla g_{t-r}(\omega^t(r)),\ u_r dB_r \rangle_{\omega^t(r)} v(r) \text{ a.s.} \tag{1.26}$$

Hence we have:

$$h(y(s)) = h(y(0)) + M(s) \text{ a.s.} \tag{1.27}$$

where $\quad M(s) = \int_0^s v(r) \langle \nabla g_{t-r}(\omega^t(r)), u_r dB_r \rangle_{\omega^t(r)}$ (1.28)

is a local martingale.

Now, we have:

$$h(y(s)) = h(t-s,\ \omega^t(s), v(s))$$

$$= (Q_N^{M_0}(t-s, t-s)f)(\omega^t(s))v(s)$$

$$= f(\omega^t(s))v(s) \tag{1.29}$$

and $h(y(0)) = h(t,x,1) = (Q_N^{M_0}(t,t-s)f)(x).$ (1.30)

Hence both $h(y(s))$ and $h(y(0))$ are bounded on Ω and hence $M(s)$ is a bounded local martingale and so is a martingale.

Consequently, we have:

$$E_x(h(y(s))) = E_x(h(y(0))) = h(y(0))$$

$$= (Q_N^{M_0}(t,t-s)f)(x) \tag{1.31}$$

i.e. $(Q_N^{M_0}(t,t-s)f)(x) = E_x(h(y(s)))$

$$= E_x(f(\omega^t(s))v(s)) \tag{1.32}$$

$$= E_x(\chi_{\zeta>s}f(\omega^t(s))v(s)) + E_x(f(\omega^t(s))v(s)\chi_{\zeta\leq s}) \tag{1.33}$$

$$= E_x(\chi_{\zeta>s}f(x^t(s))\exp\{\int_0^s \frac{LC_N}{C_N}(x^t(s))dr\}) \tag{1.34}$$

since $f(x^t(s)) = 0$ for $s \geq \zeta$. The proposition is thus proved.

(1.4) Corollary

If M has no cut-focal points with respect to N, then

$$(Q_N^M(t,t-s)f)(x) = E_x(f(x^t(s))\exp\{\int_0^s \frac{LC_N}{C_N}(x^t(u))du\}).$$

Proof.

The proof is immediate since $\zeta = +\infty$ in this case.

We can recover the Integral Formula of Chapter II from the above Proposition. First we prove:

(1.5) Lemma

$$\lim_{s\uparrow t} (Q_N^{M_o}(t,t-s)f)(x) = q_t(x,N)^{-1} \int_N f(y)p_t^{M_o}(x,y)dy.$$

Proof.

$$(Q_N^{M_o}(t,t-s)f)(x) = q_t(x,N)^{-1}P_s^{M_o}(q_{t-s}(-,N)f)(x) \tag{1.35}$$

$$P_s^{M_o}(q_{t-s}(-,N)f)(x) = \int_{M_o} q_{t-s}(z,N)f(z)p_s^{M_o}(x,z)dz \tag{1.36}$$

$$= \int_{E_\nu^o} q_{t-s}(\exp_\nu(y,v),N)f(\exp_\nu(y,v))\cdot p_s^{M_o}(x,\exp_\nu(y,v))\theta_N(y,v)dydv \tag{1.37}$$

Now, $E_\nu^o \subset N \times R^{n-k}$ and f has compact support in $M_o = \exp_\nu(E_\nu^o)$ and so (1.37) becomes after setting t-s = r:

$$(2\pi r)^{-\frac{n-k}{2}}\int_{N\times R^{n-k}}\exp\{-\frac{\|v\|^2}{2r}\}C_N(\exp_\nu(y,v))f(\exp_\nu(y,v))p_{t-r}^{M_o}(x,\exp_\nu(y,v))\theta_N(y,v)$$
$$dydv. \tag{1.38}$$

We next set $V = \sqrt{r}\,\omega$ and so (1.38) becomes:

$$(2\pi)^{-\frac{n-k}{2}} \int_N \int_{R^{n-k}} \exp\{-\frac{\|\omega\|^2}{2}\} C_N(\exp_\nu(y,\sqrt{r}\,\omega))\cdot f(\exp_\nu(y,\sqrt{r}\,\omega)) p_{t-r}^{M_0}(\exp_\nu(y,\sqrt{r}\,\omega))$$

$$\times\, \theta_N(y,\sqrt{r}\,\omega)dyd\omega \qquad (1.39)$$

which converges as $r \downarrow 0$ to:

$$(2\pi)^{-\frac{n-k}{2}} \int_N C_N(y)f(y)\theta_N(y,0)p_t^{M_0}(x,y)dy \int_{R^{n-k}} \exp\{-\frac{\|\omega\|^2}{2}\}d\omega \qquad (1.40)$$

$$= \int_N f(y)p_t^{M_0}(x,y)dy \qquad (1.41)$$

since $C_N(y) = 1 = \theta_N(y,0)$

and $\int_{R^{n-k}} \exp\{-\frac{\|\omega\|^2}{2}\}\, d\omega = (2\pi)^{\frac{n-k}{2}}$. (1.42)

The Lemma is thus proved.

(1.6) <u>Theorem</u>

(i) For N compact, we have:

$$\int_N f(y)p_t^{M_0}(x,y)dy = q_t(x,N)E_x(\chi_{\zeta>t} f(y_t^0)\exp\{\int_0^t \frac{LC_N}{C_N}(x^t(s))ds\}).$$

(ii) For any complete submanifold N, we have:

$$\int_N f(y)p_t^{M_0}(x,y)dy = q_t(x,N)E_x(\chi_{\zeta>t} f(y_t^0)\exp\{\int_0^t \frac{LC_N}{C_N}(x^t(s))ds\}).$$

Proof.

(i) We first assume that M_o is compact.

Then we let $s \uparrow t$ in Proposition (1.3): The integrand on the R.H.S. is bounded by a constant $C(t,x,N)$ depending only on (t,x,N) and so by the bounded convergence Theorem, we have:

$$\lim_{s\uparrow t} E_x(\chi_{\zeta>s} f(x^t(s))\exp\{\int_0^s \frac{LC_N}{C_N}(x^t(u))du\}) \tag{1.43}$$

$$= E_x(\chi_{\zeta>t} \lim_{s\uparrow t} f(x^t(s))\exp\{\int_0^t \frac{LC_N}{C_N}(x^t(u))du\}). \tag{1.44}$$

Recall (see Lemma (3.3) of Chapter II) that, in the case that N is compact,

$$x^t(s) = \exp_\nu(y_s^o, v_s) \tag{1.45}$$

where both $(y_s^o)_{0\le s\le t}$ and $(v_s)_{0\le s\le t}$ (with $v_t = 0$ by definition) are sample continuous (a.s.) and hence $\lim_{s\uparrow t} x^t(s) = y_t^o$ a.s.

Consequently (1.44) equals:

$$E_x(\chi_{\zeta>t} f(y_t^o)\exp\{\int_0^t \frac{LC_N}{C_N}(x^t(s))ds\}). \tag{1.46}$$

For a more general M_o, we let $(M_o^j)_{j\ge1}$ be a sequence of compact subsets of M_o exhausting M_o. We can take the sequence such that $N \subset M_o^j$ for all $j \ge 1$. We also suppose that f has support which is compact in M_o^1. By the above, we have:

$$\int_N f(y)p_t^{M_o^j}(x,y)dy = q_t(x,N)E_x(\chi_{\zeta_j>t}f(y_t^o)\exp\{\int_0^t \frac{LC_N}{C_N}(x^t(u))du\}). \quad (1.47)$$

Then taking limits as $j \uparrow \infty$, we obtain the result.

(ii) Again we take a sequence $(M_o^j)_{j\geq 1}$ of compact subsets of M_o exhausting M_o. By hypothesis N is complete in M and hence closed. Consequently $N \cap M_o^j$ is compact in M and so by (i) we have:

$$\int_{N\cap M_o^j} f(y)p_t^{M_o^j}(x,y)dy = q_t(x,N\cap M_o^j)E_x(\chi_{\zeta_j>t}f(y_t^o)\exp\{\int_0^t \frac{LC_N}{C_N}(x^t(u))du\}) \quad (1.48)$$

since $C_{N\cap M_o^j}(x) = C_N(x)$ for $x \in M_o^j$.

Then taking limits as $j \uparrow \infty$ in (1.48), we obtain the result since $N \cap M_o^j \uparrow N$ as $j \uparrow \infty$.

§2. EXACT AND ASYMPTOTIC EXPANSIONS

We will begin this section with the following important Lemma:

(2.1) Lemma

$$\frac{\partial}{\partial s}(Q_N^{M_o}(t,t-s)f)(x) = (Q_N^{M_o}(t,t-s)[\frac{L(C_N f)}{C_N} - \frac{1}{2(t-s)}\langle\nabla\rho^2,\nabla f\rangle])(x).$$

Proof.

$$(Q_N^{M_0}(t,t-s)f)(x) = q_t(x,N)^{-1}\, P_s^{M_0}(q_{t-s}(-,N)f)(x). \tag{2.1}$$

Hence we have:

$$\frac{\partial}{\partial s}(Q_N^{M_0}(t,t-s)f)(x) = q_t(x,N)^{-1}\, \frac{\partial}{\partial s}\,(P_s^{M_0}(q_{t-s}(-,N)f)(x)). \tag{2.2}$$

By definition,

$$\frac{\partial}{\partial s}(P_s^{M_0}(q_{t-s}(-,N)f)(x)) = \lim_{h\to 0} \frac{P_{s+h}^{M_0}(q_{t-(s+h)}(-,N)f)-P_s^{M_0}(q_{t-s}(-,N)f)}{h} \tag{2.3}$$

$$= \lim_{h\to 0} \frac{P_{s+h}^{M_0}(q_{t-s}(-,N)f) - P_s^{M_0}(q_{t-s}(-,N)f)}{h} \qquad \text{(I)}$$

$$\tag{2.4}$$

$$+ \lim_{h\to 0} \frac{P_{s+h}^{M_0}(q_{t-(s+h)}(-N,f) - P_{s+h}^{M_0}(q_{t-s}(-,N)f)}{h} \qquad \text{(II)}$$

Now, $\text{(I)} = \frac{\partial}{\partial r}\,(P_r^{M_0}(q_{t-s}(-,N)f))|_{r=s}$ (2.5)

$$= P_s^{M_0}(L(q_{t-s}(-,N)f)) \tag{2.6}$$

by ([4], (2) of Theorem (1.6))

$$\text{(II)} = \lim_{h\to 0} \frac{P_{s+h}^{M_0}(q_{t-(s+h)}(-,N)f) - P_{s+h}^{M_0}(q_{t-s}(-,N)f)}{h} \tag{2.7}$$

$$= P_s^{M_0}[\lim_{h\to 0} \frac{P_h^{M_0}(q_{t-(s+h)}(-,N)f) - P_h^{M_0}(q_{t-s}(-,N)f)}{h}] \tag{2.8}$$

$$= P_s^{M_o}(\lim_{h\downarrow 0} P_h^{M_o}(\frac{1}{h}(q_{t-(s+h)}(-,N)f - q_{t-s}(-,N)f))) \tag{2.9}$$

$$= P_s^{M_o}(P_o^{M_o}(\frac{\partial}{\partial s}(q_{t-s}(-,N)f))) \tag{2.10}$$

$$= P_s^{M_o}(\frac{\partial}{\partial s}(q_{t-s}(-,N)f)). \tag{2.11}$$

Set $r = t-s$, then

$$\frac{\partial}{\partial s}(q_{t-s}(-,N)f) = -\frac{\partial}{\partial r}(q_r(-,N)f) \tag{2.12}$$

$$= -\frac{\partial q_r}{\partial r}(-,N)f \tag{2.13}$$

$$= -L(q_r(-,N))f + \frac{LC_N}{C_N} q_r(-,N)f \tag{2.14}$$

$$= -L(q_{t-s}(-,N))f + \frac{LC_N}{C_N} q_{t-s}(-,N)f. \tag{2.15}$$

Therefore,

$$(II) = P_s^{M_o}[-L(q_{t-s}(-,N)f + \frac{LC_N}{C_N} q_{t-s}(-,N)f]. \tag{2.16}$$

Consequently by (2.6) and (2.16) above,

$$\frac{\partial}{\partial s}(Q_N^{M_o}(t,t-s)f)(x)$$

$$= q_t(x,N)^{-1}P_s^{M_o}[L(q_{t-s}(-,N)f)-L(q_{t-s}(-,N))f + \frac{LC_N}{C_N} q_{t-s}(-,N)f](x) \tag{2.17}$$

$$= Q_N^{M_o}(t,t-s)[\frac{L(q_{t-s}(-,N)f)}{q_{t-s}(-,N)} - \frac{L(q_{t-s}(-,N))f}{q_t(-,N)} + \frac{LC_N}{C_N} f](x). \tag{2.18}$$

Now,

$$L(q_{t-s}(-,N)f) = L(q_{t-s}(-,N))f + q_{t-s}(-,N)Lf + \langle\nabla q_{t-s}(-,N),\nabla f\rangle$$
$$- Vq_{t-s}(-,N)\cdot f. \tag{2.19}$$

Therefore (2.18) becomes:

$$Q_N^{M_o}(t,t-s)[Lf + \frac{LC_Nf}{C_N} + \langle\nabla \ \log q_{t-s}(-,N),\nabla f\rangle - Vf](x) \tag{2.20}$$

$$= Q_N^{M_o}(t,t-s)[\frac{L(C_Nf)}{C_N} - \langle\nabla \log C_N, \nabla f\rangle + \langle\nabla \log q_{t-s}(-,N),\nabla f\rangle](x) \tag{2.21}$$

Now,

$$\nabla \log q_{t-s}(x,N) = - \frac{\nabla\rho^2(x)}{2(t-s)} + \nabla \log C_N(x) \tag{2.22}$$

and so (2.21) becomes:

$$Q_N^{M_o}(t,t-s)[\frac{L(C_Nf)}{C_N} - \frac{1}{2(t-s)} \langle\nabla\rho^2,\nabla f\rangle](x) \tag{2.23}$$

and so the Lemma is proved.

For $r \geq s > 0$, define the operators $(F(r,s))$ on measurable functions f as follows:

$$(F(r,s)f)(x) = f(\gamma_N(r-s)) \tag{2.24}$$

where γ_N is the unique minimal geodesic from $x \in M_o$ to $y \in N$ in time r. γ_N is thus given in (Cartesian) Fermi coordinates as:

$$\gamma_N(u) = \sum_{j=1}^{k} x_j(x)\left(\frac{\partial}{\partial x_j}\right)_x + \left(1 - \frac{u}{r}\right) \sum_{j=k+1}^{n} x_j(x)\left(\frac{\partial}{\partial x_j}\right)_x . \tag{2.25}$$

Thus we have:

$$\gamma_N(0) = \sum_{j=1}^{n} x_j(x)\left(\frac{\partial}{\partial x_j}\right)_x = x \in M_o$$

$$\gamma_N(r) = \sum_{j=1}^{k} x_j(x)\left(\frac{\partial}{\partial x_j}\right)_x = y \in N$$

$$\gamma_N'(u) = -\frac{1}{r} \sum_{j=k+1}^{n} x_j(x)\left(\frac{\partial}{\partial x_j}\right)_x .$$

(2.2) Lemma

For $t \geq r \geq s > 0$,

$$\frac{\partial}{\partial s}(Q_N^{M_o}(t,t-s)F(t-s,t-r)f)(x) = (Q_N^{M_o}(t,t-s) \frac{L(C_N F(t-s,t-r)f))}{C_N}(x).$$

Proof.

The proof is exactly as in the previous Lemma except that (2.12) becomes:

$$\frac{\partial}{\partial s}(q_{t-s}(-,N)F(t-s,t-r)f) = -\frac{\partial q_u}{\partial u}(-,N)\cdot F(u,t-r)f_{|u=t-s} + q_u(-,N)\cdot\frac{\partial}{\partial u}(F(u,t-r)f)_{|u=t-s} \tag{2.26}$$

$$= -L(q_u(-,N))F(u,t-r)f_{|u=t-s} + \frac{LC_N}{C_N}\cdot q_u(-,N)F(u,t-r)f_{|u=t-s} + q_u(-,N)\frac{\partial}{\partial u}(F(u,t-r)f)_{|u=t-s} \tag{2.27}$$

$$= - L(q_{t-s}(-,N))F(t-s,t-r)f + \frac{LC_N}{C_N} \cdot q_{t-s}(-,N)F(t-s,t-r)f$$

$$+ q_{t-s}(-,N) \frac{\partial}{\partial u} (F(u,t-r)f)_{|u=t-s} \tag{2.28}$$

Consequently we have:

$$\frac{\partial}{\partial s} (Q_N^{M_o}(t,t-s)F(t-s,t-r)f)(x)$$

$$= Q_N^{M_o}(t,t-s)[-\frac{L(C_N F(t-s,t-r)f)}{C_N} - \frac{1}{2(t-s)} \langle \nabla\rho^2, \nabla F(t-s,t-r)f\rangle$$

$$- \frac{d}{du} (F(u,t-r)f)_{|u=t-s}] \tag{2.29}$$

Now,

$$\frac{d}{du}(F(u,t-r)f)_{|u=t-s} = \frac{d}{du} (f(\gamma_N(u-t+r))_{|u=t-s} \tag{2.30}$$

where γ_N is the unique minimal geodesic from $x \in M_o$ to $y \in N$ in time u.

We know that:

$$\gamma_N(r) = A + (1- \frac{r}{u})B \text{ where } A = \sum_{\alpha=1}^{k} x_\alpha(x)(\frac{\partial}{\partial x_\alpha})_x;\ B = \sum_{\alpha=k+1}^{n} x_\alpha(s)(\frac{\partial}{\partial x_\alpha})_x \tag{2.31}$$

Hence, $\frac{d}{du}(f(\gamma_N(u-t+r)))_{|u=t-s} = \langle \gamma_N'(u-t+r)_{|u=t-s}, \nabla f(\gamma_N(u-t+r))_{|u=t-s}\rangle$ (2.32)

$$= -\frac{t-r}{(t-s)^2}\langle B, \nabla f(\gamma_N(r-s))\rangle = - \frac{1}{2(t-s)}\langle \nabla\rho^2(x), \nabla[F(t-s,t-r)f](x)\rangle \tag{2.33}$$

since $B = \frac{1}{2}\nabla\rho^2(x)$ and $Df\circ\gamma_N(r-s)(B) = Df(\gamma_N(r-s))(B)\frac{t-r}{t-s}$. (2.34)

Thus (2.29) becomes:

$$\frac{\partial}{\partial s}(Q_N^{M_o}(t,t-s)F(t-s,t-r)f)(x) = Q_N^{M_o}(t,t-s)\left[\frac{L(C_N F(t-s,t-r)f}{C_N}\right](x) \quad (2.35)$$

and so the Lemma is proved.

(2.3) Proposition

Let γ_N be the unique minimal geodesic from $x \in M_o$ to N in time t.

Then:

$$(Q_N^{M_o}(t,t-s)f)(x) = f(\gamma_N(s)) + \sum_{j=1}^{q} a_j(s,x,N) + R_{q+1}(s,x,N)$$

where for $L_{C_N} g = \dfrac{L(C_N g)}{C_N}$,

$$a_n(s,x,N) = \int_0^s \int_0^{s_1} \cdots \int_0^{s_{n-1}} (F(t,t-s_n)L_{C_N}F(t-s_n,t-s_{n-1})L_{C_N}F(t-s_{n-1},t-s_{n-2})\cdots$$

$$\cdots L_{C_N}F(t-s_2,t-s_1)L_{C_N}F(t-s_1,t-s)f)(x)ds_1\ldots ds_n$$

for $1 \le n \le q$,

$$F_{q+1}(s,x,N) = \int_0^s \int_0^{s_1} \cdots \int_0^{q} (Q_N^{M_o}(t,t-s_{q+1})L_{C_N}F(t-s_{q+1},t-s_q) \cdots$$

$$\cdots L_{C_N}F(t-s_2,t-s_1)L_{C_N}F(t-s_1,t-s)f)(x)ds_1\ldots ds_{q+1}.$$

Proof

By Lemma (2.2) above we have for $t \ge s > 0$,

$$\int_0^s \frac{\partial}{\partial s_1}((Q_N^{M_0}(t,t-s_1)F(t-s_1,t-s)f)(x)ds_1$$

$$= \int_0^s (Q_N^{M_0}(t,t-s_1)L_{C_N}F(t-s_1,t-s)f)(x)ds_1. \tag{2.36}$$

The L.H.S. of (2.36) is given by:

$$(Q_N^{M_0}(t,t-s)F(t-s,t-s)f)(x) - (Q_N^{M_0}(t,t)F(t,t-s)f)(x). \tag{2.37}$$

Since $F(t-s,t-s)$ and $Q_N^{M_0}(t,t)$ are identity operators, (2.37) becomes:

$$(Q_N^{M_0}(t,t-s)f)(x) - (F(t,t-s)f)(x) \tag{2.38}$$

$$= (Q_N^{M_0}(t,t-s)f)(x) - f(\gamma_N(s)). \tag{2.39}$$

Thus we have:

$$(Q_N^{M_0}(t,t-s)f)(x) = f(\gamma_N(s)) + \int_0^s (Q_N^{M_0}(t,t-s_1)L_{C_N}F(t-s_1,t-s)f)(x)ds_1. \tag{2.40}$$

Set $f_1 = L_{C_N}F(t-s_1,t-s)f$. Then f_1 is smooth with compact support in M_0.
Hence for $t \geq s > 0$,

$$(Q_N^{M_0}(t,t-s_1)f_1)(x) = (F(t,t-s_1)f_1)(x) + \int_0^{s_1} (Q_N^{M_0}(t,t-s_2)L_{C_N}F(t-s_2,t-s_1)f_1(x)ds_2 \tag{2.41}$$

and so (2.40) becomes:

$$(Q_N^{M_o}(t,t-s)f)(x) = f(\gamma_N(s)) + \int_0^s (F(t,t-s_1)L_{C_N}F(t-s_1,t-s)f)(x)ds_1$$

$$+ \int_0^s \int_0^{s_1} (Q_N^{M_o}(t,t-s_2)L_{C_N}F(t-s_2,t-s_1)L_{C_N}F(t-s_1,t-s)f)(x)ds_1ds_2. \tag{2.42}$$

The Proposition is thus proved for $n = 1$. Suppose by induction that it is true for n; $1 \leq n \leq q$. Set

$$f_n = L_{C_N}F(t-s_n,t-s_{n-1})L_{C_N}(t-s_{n-1},t-s_{n-2}) \ldots L_{C_N}F(t-s_1,t-s)f.$$

Then computations as before show that:

$$\int_0^s \int_0^{s_1} \ldots \int_0^{s_{n-1}} (Q_N^{M_o}(t,t-s_n)f_n)(x) = a_n(s,x,N) + F_{n+1}(s,x,N) \tag{2.43}$$

i.e. $F_n(s,x,N) = a_n(s,x,N) + F_{n+1}(s,x,N)$

and so the Proposition is proved.

A natural assumption for the Theorem below is that f depends only on the projection of M_o onto N (viewed in Cartesian Fermi Coordinates).

(2.4) <u>Theorem</u> (An Exact Expansion Formula)

Let N be a complete and f smooth of compact support in M_o and depending only on the projection of M_o onto N. Then for $\frac{LC_N}{CN}$ and $L_{C_N}F(1-r_{q+1},1-r_q) \ldots L_{C_N}F(1-r_2,1-r_1)L_{C_N}{}^{f\circ p_N}$ bounded, we have:

$$\int_N f(y)p_t^{M_o}(x,y)dy = q_t(x,N)[f(\gamma_N(t)) + \sum_{j=1}^{q} b_j(x,N)t^j + R_{q+1}(t,x,N)t^{q+1}]$$

for all $q \geq 1$ where

$$b_1(x,N) = \int_0^1 (F(1,1-r_1)L_{C_N} f \circ p_N)(x)dr_1$$

$$b_n(x,N) = \int_0^1 \int_0^{r_1} \cdots \int_0^{r_{n-1}} (F(1,1-r_n)L_{C_N} F(1-r_n,1-r_{n-1}) \cdots$$

$$\cdots L_{C_N} F(1-r_2,1-r_1)L_{C_N} f \circ p_N)(x)dr_1 \ldots dr_n$$

for $2 \leq n \leq q$,

$$R_{q+1}(t,x,N) = \int_0^1 \int_0^{r_1} \cdots \int_0^{r_q} (Q_N^{M_o}(t,t-tr_{q+1})L_{C_N} F(1-r_{q+1},1-r_q)$$

$$L_{C_N} F(1-r_q,1-r_{q-1}) \ldots L_{C_N} F(1-r_2,1-r_1)L_{C_N} f \circ p_N)(x)dr_1 \ldots dr_{q+1}$$

$$= \int_0^1 \int_0^{r_1} \cdots \int_0^{r_q} E_x(\chi_{\zeta > tr_{q+1}}(L_{C_N} F(1-r_{q+1},1-r_q)L_{C_N} F(1-r_q,1-r_{q-1}) \cdots$$

$$\cdots L_{C_N} F(1-r_2,1-r_1)L_{C_N} f \circ p_N)(x^t(tr_{q+1}))\exp\{\int_0^{tr_{q+1}} \frac{LC_N}{C_N}(x^t(s))ds\})dr_1 \ldots dr_{q+1}$$

where the integrand is assumed bounded and $p_N : M_o \longrightarrow N$ is the projection of M_o onto N (viewed in Fermi Coordinates).

<u>Proof</u>

Consider $(F(t-s_1,t-s)f(x) = f(\gamma_N(s-s_1))$

where γ_N is the unique minimal geodesic from x to N in time $t-s_1$: In Cartesian Fermi-Coordinates,

$$\gamma_N(s-s_1) = \sum_{j=1}^{k} x_j(x)\left(\frac{\partial}{\partial x_j}\right)_x + \left(1 - \frac{s-s_1}{t-s_1}\right) \sum_{j=k+1}^{m} x_j(x)\left(\frac{\partial}{\partial x_j}\right)_x \tag{2.44}$$

Since f depends only on the projection of M_o onto N, we have:

$$f(\gamma_N(s-s_1)) = f(\gamma_N(t-s_1)). \tag{2.45}$$

Also, $\gamma_N(t-s_1) = p_N(\gamma_N(0)) = p_N(x)$. Thus, $f(\gamma_N(t-s_1)) = f\circ p_N(x)$ and so,

$$(F(t-s_1,t-s)f)(x) = f\circ p_N(x). \tag{2.46}$$

We now let $s \uparrow t$ in Proposition (2.3):

For the L.H.S,

$$\lim_{s\uparrow t} (Q_N^{M_o}(t,t-s)(x) = q_t(x,N)^{-1} \int_N f(y)p_t^{M_o}(x,y)dy$$

by Lemma (1.5). For the R.H.S, set $s_i = tr_i$; $i = 1,2,\dots,n$. Then,

$$F(t,t-s_n) = F(t,t-tr_n) = F(1, \frac{t-tr_n}{t}) = F(1,1-r_n)$$

$$F(t-s_i,t-s_{i-1}) = F(t-tr_i,t-tr_{i-1}) = F(1, \frac{t-tr_{i-1}}{t-tr_i})$$

$$= F(1, \frac{1-r_{i-1}}{1-r_i}) = F(1-r_i, 1-r_{i-1})$$

for $i = 2,\dots,n$. Consequently in Proposition (2.3),

$$a_n(s,x,N) = \int_0^{s/t}\int_0^{r_1}\cdots\int_0^{r_{n-1}} (F(1,1-r_n)L_{C_N}F(1-r_n,1-r_{n-1})L_{C_N}F(1-r_{n-1},1-r_{n-2})\cdots$$

$$\cdots L_{C_N}F(1-r_2,1-r_1)L_{C_N}f\circ p_N)(x)t^n\, dr_1 \dots dr_n. \tag{2.47}$$

Set $f_{r_1,\dots,r_n} = L_{C_N}F(1-r_n,1-r_{n-1})\dots L_{C_N}F(1-r_2,1-r_1)L_{C_N}{}^{f\circ p_N}$.

Then $f_{r_1,\dots,r_n}$ is bounded by a constant independent of $r_1,\dots,r_n$ and hence,

$$a_n(s,x,N) \to b_n(x,N)t^n \text{ as } s \uparrow t.$$

Similarly we have:

$$F_{q+1}(s,x,N) = \int_0^{s/t}\int_0^{r_1}\dots\int_0^{r_q} E_x(\chi_{\zeta>tr_{q+1}} f_{r_1,\dots,r_{q+1}}(x^t(tr_{q+1}))$$

$$\exp\{\int_0^{tr_{q+1}} \frac{LC_N}{C_N}(x^t(u))du\}dr_1\dots dr_{q+1}. \tag{2.48}$$

Since the integrand is assumed bounded, it is clear that:

$$F_{q+1}(s,x,N) \to R_{q+1}(t,x,N)t^{q+1} \text{ as } s \uparrow t \tag{2.49}$$

and so the Theorem is proved.

(2.5) <u>Theorem</u> (Asymptotic Expansion Formula)

Suppose that the integrand above is bounded i.e. both $f_{r_1,\dots,r_{q+1}}$ and $\frac{LC_N}{C_N}$ are bounded. Then:

$$\int_N f(y)p_t^{M_O}(x,y)dy = q_t(x,N)[f(\gamma_N(t)) + \sum_{j=1}^{q} b_j(x,N)t^j + o(t^q)]$$

Proof

It is clear that $R_{q+1}(t,x,N)$ is bounded by the above boundedness assumption and hence

$$R_{q+1}(t,x,N)t^{q+1} = o(t^q).$$

§3. SOME APPLICATIONS-SUBMERSIONS AND HEAT KERNELS

Let $\pi:M \to B$ be a submersion (not necessarily Riemannian). This means that the derivative map $T\pi$ is such that:

$$T_x\pi : T_xM \longrightarrow T_{\pi(x)}B$$

is surjective for each $x \in M$.

(3.1) Proposition

$$\int_{\pi^{-1}(q)} f(y)p_t^{M_0}(x,y)dy = (2\pi t)^{-\frac{b}{2}} C_N(x)\exp\{-\frac{d(x,N)^2}{2t}\}$$

$$\times \; E_x(\chi_{\zeta>t}f(y_t^0)\exp\{\int_0^t -\frac{LC_N}{C_N}(x^t(s))ds\})$$

where M_o is a tubular neighbourhood of $N = \pi^{-1}(q)$ in M and b = dimension of B.

Proof.

This is immediate from the Integral Formula (Chapter II, Theorem 4.3 and its Corollary (4.6)). The only difference is that the factor $(2\pi t)^{-\frac{n-k}{2}}$ is replaced by $(2\pi t)^{-\frac{b}{2}}$. This is because by ([10], Theorem 5.8 p. 79), $N = \pi^{-1}(q)$ is a (closed, regular) submanifold of dimension $n-b$ where b = rank of $\pi = \dim T_x\pi(T_xM) = \dim B$. The last equality is due to the surjectivity of $T_x\pi$ for each $x \in M$.

We shall suppose that the set of cut-focal points of M with respect $N = \pi^{-1}(q)$ has capacity zero. Hence we have:

$$p_t^M(x,y) = p_t^{M_o}(x,y) \text{ for all } (x,y) \in M_o \times M_o. \tag{3.1}$$

If we choose $f = \sqrt{\det(T.\pi)(T.\pi)^*}^{-1}$, then

$$\int_{\pi^{-1}(q)} \sqrt{\det(T_y\pi)(T_y\pi)^*}\, p_t^M(x,y)dy = (2\pi t)^{-\frac{b}{2}} C_N(x)\exp\{-\frac{d(x,N)^2}{2t}\}$$
$$\times\ E_x(\chi_{\zeta>t} \sqrt{\det(T_{y_t}o\pi)(T_{y_t}o\pi)^*} \exp\{\int_0^t -\frac{LC_N}{C_N}(x^t(s))ds\}). \tag{3.2}$$

The following is due to K.D. Elworthy (see also [9] pages 1-3). The proof is left out.

(3.2) Theorem

Let M and B be Riemannian manifolds and let $\pi: M \to B$ be a submersion.

Let $(z_t)_{0 \leq t < +\infty}$ be a diffusion on M with Heat Kernel $p_t^M(-,-)$ and let $\rho_t : B \longrightarrow \mathbb{R}$ be the density of $\pi(z_t)$. Then we have:

$$\rho_t(q) = \int_{\pi^{-1}(q)} \sqrt{\det(T_y\pi)(T_y\pi)^*}^{\,-1}\, p_t^M(x,y)\,dy.$$

(3.3) <u>Theorem</u>

Under the conditions of Theorem (3.2), the density $\rho_t(q)$ has the following expansion:

$$\rho_t(q) = (2\pi t)^{-\frac{b}{2}}\, C_N(x) \exp\{-\frac{d(x,N)^2}{2t}\} [\cancel{1}\, b_0 + b_1(x,N)t$$

$$+ b_2(x,N)t^2 + \ldots + b_m(x,N)t^m + R_{m+1}(t,x,N)t^{m+1}]; \quad b_0 = \sqrt{\det(T_{\gamma_N(t)}\pi)(T_{\gamma_N(t)}\pi)^*}^{\,-1}$$

<u>Proof</u>.

The proof is immediate by the Exact Expansion Formula (Theorem (2.4)) and Theorem (3.2) above.

(3.4) <u>Remarks</u>

(i) We are (of course) assuming suitable boundedness conditions as in Theorem (2.5).

(ii) We can thus replace $R_{m+1}(t,x,N)t^{m+1}$ by $o(t^m)$.

(iii) In particular if $B = \mathbb{R}^b$, then

$$\rho_{t'}(q) = \int_{\Pi^{-1}(q)} \sqrt{\det(D\Pi(y))(D\Pi(y))^*}\, p_t^M(x,y)dy \tag{3.3}$$

Thus $\rho_t\,(q)$ depends on Π and its derivatives along $N = \Pi^{-1}(q)$.

(iv) If the submersion $\Pi : M \to B$ is Riemannian, then:

$(T_y\Pi)(T_y\Pi)^* : T_{\Pi(y)}B \to T_{\Pi(y)}B$ is the identity and so:

$$\rho_t(p,q) = \int_{\Pi^{-1}(q)} p_t^M(x,y)dy \tag{3.4}$$

where $p = \Pi(x)$. This is just the formula of Theorem (6.1) of Chapter II.

(v) When $B = R^b$ and the fibres $\Pi^{-1}(q)$ are minimal submanifolds of M, then by ([14], Chapter IX, Theorem 10E), $\Pi(z_t)$ is isonomous to Brownian Motion on R^b since $(z_t)0 \le t < +\infty$ is Brownian Motion on M with $z_o = x$. Consequently,

$$\int_{\Pi^{-1}(q)} p_t^M(x,y)dy = \rho_t(p,q) = (2\Pi t)^{-\frac{b}{2}} \exp\{- \frac{\|p-q\|^2}{2t}\} \tag{3.5}$$

where $p = \Pi(x)$.

Also, we have:

$$\begin{aligned} \int_{\Pi^{-1}(q)} p_t^M(x,y)dy = {} & (2\Pi t)^{-\frac{b}{2}} C_N(x)\exp\{- \frac{d(x,N)^2}{2t}\} \\ & \times E_x(\chi_{\zeta>t} \exp\{\int_0^t \frac{LC_N}{C_N}(x^t(s))ds\}) \end{aligned} \tag{3.6}$$

$$= q_t(x,N)[1 + b_1(x,N)t + \ldots + b_m(x,N)t^m + o(t^m)] \tag{3.7}$$

by the Asymptotic Expansion Formula.

Thus we have the equalities:

$$(2\pi t)^{-\frac{b}{2}} \exp\{-\frac{\|p-q\|^2}{2t}\}$$

$$= (2\pi t)^{-\frac{b}{2}} C_N(x)\exp\{-\frac{d(x,N)^2}{2t}\} E_x(\chi_{\zeta>t} \exp\{\int_0^t \frac{LC_N}{C_N}(x^t(s)ds\}) \qquad (3.8)$$

$$= (2\pi t)^{-\frac{b}{2}} C_N(x)\exp\{-\frac{d(x,N)^2}{2t}\} [1 + \sum_{j=1}^{m} b_j(x,N)t^j + o(t^m)] \qquad (3.9)$$

In this case $d(x,N) = \|p-q\|$, the equalities of (3.8) and (3.9) become:

$$1 = C_N(x)E_x(\chi_{\zeta>t} \exp\{\int_0^t \frac{LC_N}{C_N}(x^t(s))ds\}$$

$$= C_N[1 + \sum_{j=1}^{m} b_j(x,N)t^j + o(t^m)]$$

whence:

$$C_N(x) = 1 = E_x(\chi_{\zeta>t} \exp\{\int_0^t \frac{LC_N}{C_N}(x^t(s))ds\})$$

$$b_j(x,N) = 0 \text{ for all } j = 1,2,3,...$$

CHAPTER IV : EXACT AND ASYMPTOTIC EXPANSIONS OF THE HEAT KERNEL IN A GEODESIC CHART

§0. INTRODUCTION

We will continue to assume that M is a connected complete n-dimensional Riemannian manifold and $L = \frac{1}{2}\Delta + b + V$ where b is a smooth vector field on M and V a potential term which we assume continuous at first and smooth later. $p_t^M(-,-)$ is the fundamental solution associated to L (for the existence of $p_t^M(-,-)$ see [3], Theorem 1).

Let $(P_t)_{t\geq 0}$ be the associated Markov Semigroup of operators. If y is a pole for M; K.D. Watling in [37] has defined Semi-Classical Semigroups $(Q_y(t,s))t \geq s > 0$ and their associated Semi-Classical Brownian Riemannian Bridges $(x^t(s))0 \leq s \leq t$. He then proves important properties of such Semi-Classical Semigroups leading up to: (1) an extended version of the Elementary Heat Kernel Formula of K.D. Elworthy and A. Truman (see [17]) and (2) an Exact Expansion Formula which exhibits small time asymptotic behaviour.

Here we will generalize the results in [37] to a local situation. Let $(\exp_y^{-1},U)$ be a geodesic chart at $y \in M$ where U lies in a star-shaped set and has compact closure and smooth boundary as in Chapter I. Let $p_t^U(-,-)$ be the Heat Kernel in U associated to the operator L i.e. the fundamental solution of L with Dirichlet boundary conditions (for the existence of $p_t^U(-,-)$ see [3], Theorem 2). Finally let $(P_t^U)_{t\geq 0}$ be the associated Semigroup of operators.

What we have done in Chapters II and III already provides a further generalization of the above situation: When we take $N = \{y\}$, then (1) the Fermi coordinates reduce to the usual geodesic normal coordinates about $y \in M$.

(2) The tubular neighbourhood M_o of N becomes the domain U of a geodesic chart $(\exp_y^{-1}, U)$, (3) the Heat Kernel $p_t^{M_o}(-,-)$ is replaced by $p_t^U(-,-)$ and (4) the Markov Semigroup $P_t^{M_o}$ is replaced by P_t^U. We will simply reproduce the corresponding (and relevant) results in this Chapter. These will however lead to more important results on more general manifolds (i.e. not necessarily having a pole at any point).

We will (by this method) obtain (1) a local version of the extended Elementary Formula of K.D. Elworthy and A. Truman (as we did by another method in Chapter I), (2) an Exact Expansion Formula from which we deduce an Asymptotic Expansion Formula for $p_t^U(x,y)$ and $p_t^M(x,y)$, (3) under certain conditions, formulae linking coefficients in the expansion of $p_t^M(-,-)$ and those of the integral of the Heat Kernel over a submanifold, (4) making use of Milson's formula, coefficients in the expansion of the Heat Kernel of the Hyperbolic n-space. We will further compute the coefficients $b_1(y,y)$ and $b_2(y,y)$ in the small time asymptotic expansion of $p_t^M(y,y)$ in terms of the curvature at y and we will finally give the "raw" expression for the fourth coefficient $b_3(y,y)$.

§1. NOTATIONS

The notation is as in Chapter I where the parameter $\mu = 1$ or as in Chapters II and III where $N = \{y\}$. In particular Cut(y) will denote the cut-locus of M at $y \in M$ and $(x^t(s))0 \le s \le t \wedge \zeta$ the Semi-Classical Brownian Riemannian Bridge from $x \in M \setminus Cut(y)$ to y in time t and first exiting from $M \setminus Cut(y)$ at the (random) time ζ. In other words, $(x^t(s))0 \le s \le t \wedge \zeta$ is the inhomogeneous process with generator $\frac{1}{2}\Delta + b + \nabla \log q_{t-s}(-,y)$ where

$$q_t(x,y) = (2\pi t)^{-\frac{n}{2}} C_y(x) \exp\{-\frac{d(x,y)^2}{2t}\} \quad .$$

§2. SEMI-CLASSICAL SEMIGROUPS AND EXACT FORMULA FOR THE DIRICHLET HEAT KERNEL IN U.

Set $\omega^t(r) = x^t(r \wedge \tau)$ (2.1)

where $(x^t(s))0 \leq s \leq t \wedge \zeta$ is the Semi-Classical Bridge Process defined above and $\tau = \tau(t)$ is its first exit time from $\bar{U}$. Observe that $\omega^t(r)$ is the process $x^t(r)$ started at $x \in U$ and killed at the first exit from $\bar{U}$.

Define for $t \geq s > 0$ and bounded measurable functions f the Semi-Classical Semigroups on U $(Q_y^U(t,s))$ as follows:

$$(Q_y^U(t,s)f)(x) = q_t(x,y)^{-1} P_{t-s}^U(q_s(\cdot,y)f)(x). \tag{2.2}$$

(2.1) Theorem

For f smooth and of support in U, we have for $t \geq s > 0$,

$$(Q_y^U(t,t-s)f)(x) = \mathbb{E}_x(\chi_{\tau>s}f(x^t(s))\exp\{\int_0^s \frac{LC_y}{C_y}(x^t(r))dr\}).$$

Proof.

This follows at once from Proposition (1.3) of Chapter III where we take $N = \{y\}$.

The Corollary below immediately follows as a special case of Corollary (1.4) of Chapter III.

(2.2) Corollary

If M has a pole at $y \in M$, then

$$(Q_y^M(t,t-s)f)(x) = E_x(f(x^t(s))\exp\{\int_0^s \frac{LC_y}{C_y}(x^t(r))dr\}).$$

(2.3) Remark

The above Corollary is Proposition (2.7) in [37]. Observe that we have removed the boundedness condition imposed there on $\frac{LC_y}{C_y}$.

Next we have one of the principal results of this section: the local version of the extended Elementary Formula of K.D. Elworthy and A. Truman.

First we have the Lemma below.

(2.4) Lemma

Let $U \subset M \setminus Cut(y)$ with compact closure and smooth boundary. Then we have, for $f(y) = 1$,

$$\lim_{s \uparrow t} (Q_y^U(t,t-s)f)(x) = q_t(x,y)^{-1} p_t^U(x,y).$$

Proof.

The proof is clear from the proof of Lemma (1.5) of Chapter III where we take $N = \{y\}$.

(2.5) Theorem (the Local Version of Elementary Formula)

(i) Let $U \subset M \setminus Cut(y)$ with compact closure and smooth boundary. Then, for $f(y) = 1$,

$$p_t^U(x,y) = q_t(x,y)E_x(\chi_{\tau>t}\exp\{\int_0^t -\frac{LC_y}{C_y}(x^t(r))dr\}).$$

(ii) For any $U \subset M \setminus Cut(y)$, we have, for $f(y) = 1$,

$$p_t^U(x,y) = q_t(x,y)E_x(\chi_{\tau>t}\exp\{\int_0^t \frac{LC_y}{C_y}(x^t(s))ds\}).$$

Proof.

(i) We let $s \uparrow t$ in Theorem (2.1): The integrand on the R.H.S. is bounded by a constant depending only on (t,x,y) and so by the bounded convergence Theorem,

$$\lim_{s\uparrow t} E_x(\chi_{\tau>t}f(x^t(s))\exp\{\int_0^s \frac{LC_y}{C_y}(x^t(r))dr\})$$

$$= E_x(\chi_{\tau>t}f(x^t(t))\exp\{\int_0^t \frac{LC_y}{C_y}(x^t(s))ds\}) \tag{2.3}$$

$$= E_x(\chi_{\tau>t}\exp\{\int_0^t \frac{LC_y}{C_y}(x^t(s))ds\}) \tag{2.4}$$

since $f(x^t(t)) = f(y) = 1$.

Lemma (2.4) above then gives the L.H.S.

(ii) We take an increasing sequence $(U_n)_{n\geq 1}$ of subsets of U exhausting U and such that for each $n \geq 1$, U_n is of compact closure and of smooth boundary. By (i), we have:

$$p_t^{U_n}(x,y) = q_t(x,y)E_x(\chi_{\tau^n>t}\exp\{\int_0^t \frac{LC_y}{C_y}(x^t(r))dr\}) \tag{2.5}$$

where $\tau^n = \tau^n(t)$ is the first exit time of the Semi-Classical Bridge Process $(x^t(s))0 \leq s \leq t \wedge \zeta$ from U_n. By taking limits as $n \uparrow \infty$, we have by the Monotone Convergence Theorem:

$$\lim_{n}\uparrow p_t^{U_n}(x,y) = q_t(x,y)E_x(\chi_{\tau>t} \exp\{\int_0^t \frac{LC_y}{C_y}(x^t(s))ds\}) \tag{2.6}$$

since $\lim_{n}\uparrow \tau^n = \tau$ where $\tau = \tau(t)$ is the first exit time of the Bridge process from U. By ([11], Chapter VIII, Theorem 4),

$$\lim_{n}\uparrow p_t^{U_n}(x,y) = p_t^U(x,y). \tag{2.7}$$

(2.6) Corollary

If M has a pole at $y \in M$, then

$$p_t^M(x,y) = q_t(x,y)E_x(\exp\{\int_0^t \frac{LC_y}{C_y}(x^t(s))ds\}).$$

Proof.

In (ii) of Theorem (2.5) above, we take U = M since in this case $Cut(y) = \emptyset$. The result follows by noting that $(x^t(s))0 \le s \le t \wedge \zeta$ is non-explosive i.e. $\zeta = +\infty$.

In particular, we note that the expectation in the above Corollary is finite without the boundedness condition of [37].

We will next obtain some supplementary results from Theorem (2.5):

(2.7) Theorem

If $q_{t,y}^U(-,-)$ is the fundamental solution of the Cauchy Problem with Dirichlet Boundary Conditions in U:

$$\frac{\partial f_t}{\partial t} = Lf_t - \frac{LC_y}{C_y} f_t$$

$$f_o = f$$

$$f_{t|\partial U} \equiv 0$$

i.e.

$$f_t(p) = \int_U f(z)q^U_{t,y}(p,z)dz,$$

then,

(i) $q^U_{t,y}(x,y) = q_t(x,y)P_x(\tau > t)$

(ii) $P_x(\tau > t) = 1 + o(t^N)$ for all $N \geq 1$ if $x \in B(y,\delta) \subset U$ for some $\delta > 0$.

<u>Proof</u>.

(i) An easy computation shows that:

$$Lf_t - \frac{LC_y}{C_y} f_t = L'f_t \tag{2.8}$$

$$\text{where } V' = V - \frac{LC_y}{C_y} \tag{2.9}$$

$$\text{and} \quad L' = \frac{1}{2}\Delta + b + V' \tag{2.10}$$

Hence by (ii) of Theorem (2.5),

$$q^U_{t,y}(x,y) = q_t(x,y)E_x(\chi_{\tau>t} \exp\{\int_0^t \frac{L'C_y}{C_y}(x^t(s))ds\}) \tag{2.11}$$

Now,

$$L'C_y = (\frac{1}{2}\Delta + b + V')C_y \tag{2.12}$$

$$= \frac{1}{2}\Delta C_y + \langle b,\nabla C_y\rangle + V'C_y \tag{2.13}$$

$$= \frac{1}{2}\Delta C_y + \langle b,\nabla C_y\rangle + (V - \frac{LC_y}{C_y})C_y \tag{2.14}$$

$$= \frac{1}{2}\Delta C_y + \langle b, \nabla C_y\rangle + VC_y - LC_y \tag{2.15}$$

$$= (L-V)C_y - (L-V)C_y = 0 \tag{2.16}$$

and hence

$$\frac{L'C_y}{C_y} = 0. \tag{2.17}$$

Consequently by (2.11) we have:

$$q^U_{t,y}(x,y) = q_t(x,y)P_x(\tau > t). \tag{2.18}$$

We deduce in particular that:

(a) When M has a pole at y, then $q_t(-,y)$ is the fundamental solution of the Cauchy Problem in M:

$$\begin{aligned} &\frac{\partial f_t}{\partial t} = Lf_t - \frac{LC_y}{C_y} \\ &f_o = f. \end{aligned} \tag{2.19}$$

We thus obtain Proposition (1.4) of [37].

(b) $(t,x) \to P_x(\tau > t)$ is a $C^{1,2}((0,\infty) \times U)$-function. In fact by setting

$$h_t(x) = P_x(\tau > t) \tag{2.20}$$

$$L^t = \frac{1}{2}\Delta + b + \nabla \log q_t(-,y) \tag{2.21}$$

we have:

$$\frac{\partial h_t}{\partial t} = L^t h_t \ . \tag{2.22}$$

(ii) We first assume that $M = R^n$, U is a Euclidean ball $D = D(0,\delta)$; $\delta > 0$, and $b \equiv 0$. Then

$$\frac{LC_y}{C_y} \equiv 0$$

and by the small time asymptotic expansion Theorem (Theorem (4.1)) of this Chapter,

$$p_t^D(x,0) = (2\pi t)^{-\frac{n}{2}} (1 + o(t^N)). \tag{2.23}$$

But by Theorem (2.5) above,

$$p_t^D(x,0) = (2\pi t)^{-\frac{n}{2}} P_x(\tau^D > t) \tag{2.24}$$

where $\tau^D = \tau^D(t)$ is the first exit time of the n-dimensional Euclidean Brownian Bridge from D. Hence we have:

$$P_x(\tau^D > t) = 1 + o(t^N). \tag{2.25}$$

Next suppose M and U are as before. Then for $B = B(y,\delta) \subset U$ we have:

$$P_x(\tau^B > t) \leq P_x(\tau > t). \tag{2.26}$$

But

$$P_x(\tau^B > t) = P_x(\tau^D > t) \tag{2.27}$$

since the n-dimensional Semi-Classical Brownian Riemannian Bridge and the n-dimensional Euclidean Brownian Bridge are radially the same in distribution. Consequently (2.25) and (2.27) give

$$P_x(\tau^B > t) = 1 + o(t^N). \tag{2.28}$$

Thus we have:

$$1 + o(t^N) = P_x(\tau^B > t) \le P_x(\tau > t) \le 1 \tag{2.29}$$

and this implies that

$$P_x(\tau > t) = 1 + o(t^N). \tag{2.30}$$

However if b is a non-zero Vector field, the use of Varadhan's estimate seems unavoidable: By Varadhan's estimate (see for example proof of Theorem (2.2) in [31]): there exists $\delta > 0$, such that:

$$q^U_{t,y}(x,y) \ge q^M_{t,y}(x,y) - e^{-\frac{\|x\|^2+\delta}{2t}} \tag{2.31}$$

But

$$q^M_{t,y}(x,y) = q_t(x,y) \tag{2.32}$$

and so we have:

$$q^U_{t,y}(x,y) \ge q_t(x,y) - e^{-\frac{\|x\|^2+\delta}{2t}}. \tag{2.33}$$

Hence we have:

$$1 \geq P_x(\tau > t) = q_t(x,y)^{-1} \cdot q^U_{t,y}(x,y) \tag{2.34}$$

$$\geq q_t(x,y)^{-1}(q_t(x,y) - e^{-\frac{\|x\|^2+\delta}{2t}}) \tag{2.35}$$

$$= 1 - (2\pi t)^{n/2} \, C_y(x)^{-1} \, e^{-\frac{\delta}{2t}} \tag{2.36}$$

and hence,

$$0 \leq 1 - P_x(\tau > t) \leq (2\pi t)^{n/2} \, C_y(x)^{-1} \, e^{-\frac{\delta}{2t}} \tag{2.37}$$

and so,

$$P_x(\tau(t) > t) = 1 + o(t^N). \tag{2.38}$$

§3. AN EXACT EXPANSION FORMULA FOR THE DIRICHLET HEAT KERNEL IN U

As in Chapter III, we define for $r \geq s > 0$, the operator $(F(r,s))$ on the space of measurable functions f as follows:

$$(F(r,s)f)(x) = f(\gamma(r-s))$$

where γ is the unique minimal geodesic from $x \in U$ to y in time r.

$$\gamma(s) = x - \frac{s}{r}x$$

in geodesic normal coordinates.

We will henceforth assume that the smooth function f with support in U has constant value 1 in a neighbourhood of the geodesic γ between x and y and that V is smooth.

(3.1) <u>Theorem</u>

$$(Q_y^U(t,t-s)f)(x) = 1+a_1(s,x,y) + a_2(s,x,y)+\ldots+a_N(s,x,y) + F_{N+1}(s,x,y)$$

where for s > 0

$$a_1(s,x,y) = \int_0^s (F(t,t-s_1)\frac{LC_y}{C_y})(x)ds_1 .$$

$$F_2(s,x,y) = \int_0^s \int_0^{s_1} (Q_y^U(t,t-s_2)L_{C_y}F(t-s_2,t-s_1)\frac{LC_y}{C_y})(x)ds_1 ds_2$$

and for $2 \le n \le N$,

$$a_n(s,x,y) = \int_0^s \int_0^{s_1} \cdots \int_0^{s_{n-1}} (F(t,t-s_n)L_{C_y}F(t-s_n,t-s_{n-1})$$

$$L_{C_y}F(t-s_{n-1},t-s_{n-2})\ldots L_{C_y}F(t-s_2,t-s_1)\frac{LC_y}{C_y})(x)ds_1\ldots ds_n$$

$$F_{N+1}(s,x,y) = \int_0^s \int_0^{s_1}\cdots\int_0^{s_N} (Q_y^U(t,t-s_{N+1})L_{C_y}F(t-s_{N+1},t-s_N)$$

$$L_{C_y}F(t-s_N,t-s_{N-1})\ldots L_{C_y}F(t-s_2,t-s_1)\frac{LC_y}{C_y})(x)ds_1\ldots ds_{N+1}$$

where $L_{C_y} g = \dfrac{L(C_y g)}{C_y}$.

<u>Proof.</u>

This is a special case of Proposition (2.3) of Chapter III where we take N = {y}.

(3.2) Theorem (An Exact Expansion Formula)

$$p_t^U(x,y) = q_t(x,y)[1 + b_1(x,y)t + b_2(x,y)t^2 + \ldots + b_N(x,y)t^N + R_{N+1}(t,s,y)t^{N+1}]$$

where

$$b_1(x,y) = \int_0^1 (F(1,1-r_1)\frac{LC_y}{C_y})(x)dr_1$$

$$b_n(x,y) = \int_0^1 \int_0^{r_1} \cdots \int_0^{r_{n-1}} (F(1,1-r_n)L_{C_y}F(1-r_n,1-r_{n-1})L_{C_y}F(1-r_{n-1},1-r_{n-2}) \cdots$$

$$\cdots L_{C_y}F(1-r_2,1-r_1)\frac{LC_y}{C_y})(x)dr_1dr_2\ldots dr_n$$

for $2 \le n \le N$.

$$R_{N+1}(t,x,y) = \int_0^1 \int_0^{r_1} \cdots \int_0^{r_N} (Q_y^U(t,tr_{N+1})L_{C_y}F(1-r_{N+1},1-r_N)$$

$$L_{C_y}F(1-r_N,1-r_{N-1})\ldots L_{C_y}F(1-r_2,1-r_1)\frac{LC_y}{C_y})(x)dr_1dr_2\ldots dr_{N+1}$$

$$= \int_0^1 \int_0^{r_1} \cdots \int_0^{r_N} E_x(\chi_{\tau>tr_{N+1}}(L_{C_y}F(1-r_{N+1},1-r_N)L_{C_y}F(1-r_N,1-r_{N-1}) \cdots$$

$$\cdots L_{C_y}F(1-r_2,1-r_1)\frac{LC_y}{C_y})(x^t(tr_{N+1})\exp\{\int_0^{tr_{N+1}} \frac{LC_y}{C_y}(x^t(s))ds\}dr_1\ldots dr_{N+1}.$$

Proof.

Again, this is a special case of Theorem (2.4) of Chapter III where we take $N = \{y\}$.

(3.3) <u>Corollary</u>

$$p_t^{M\setminus Cut(y)}(x,y) = q_t(x,y)[1+b_1(x,y)t+b_2(x,y)^2t+\ldots+b_N(x,y)t^N+R'_{N+1}(t,x,y)t^{N+1}]$$

where for $1 \le n \le N$, $b_n(x,y)$ is defined as in Theorem (3.2) above and $R'_{N+1}(t,x,y)$ is an improper integral that will be defined in the proof.

<u>Proof</u>.

We take an increasing sequence exhausting $M\setminus Cut(y)$ $(U_k)_{k\ge 1}$ such that each U_k is of compact closure and smooth boundary. Setting

$$f_{r_1,\ldots,r_n} = L_{C_y}F(1-r_n,1-r_{n-1})\ldots L_{C_y}F(1-r_2,1-r_1)\frac{LC_y}{C_y},$$

$$p_t^{U_k}(x,y) = q_t(x,y)[1 + \sum_{j=1}^{N} b_j(x,y)t^j + R^k_{N+1}(t,x,y)t^{N+1}] \tag{3.1}$$

with

$$R^k_{N+1}(t,x,y) = \int_0^1 \int_0^{r_1}\ldots\int_0^{r_N} E_x(\chi_{\zeta^k > tr_{N+1}} f_{r_1,\ldots,r_{N+1}}(x^t(tr_{N+1}))$$

$$\exp\{\int_0^{tr_{N+1}} \frac{LC_y}{C_y}(x^t(s))ds\}) \tag{3.2}$$

where $\zeta^k = \zeta^k(t)$ is the first exit time of $(x^t(s))0 \le s \le t \wedge \zeta$ from U_k. We take limits as $k \uparrow +\infty$:

$$\text{Now, } \lim_{k}\uparrow p_t^{U_k}(x,y) = p_t^{M\setminus Cut(y)}(x,y) \tag{3.3}$$

by ([11]; Chapter VIII, Theorem 4).

$\lim_{k\uparrow\infty} R^k_{N+1}(t,x,y)$ must exist since all other terms in (3.1) exist as $k \uparrow \infty$. We set:

$$R'_{N+1}(t,x,y) = \lim_{k\uparrow\infty} R^k_{N+1}(t,x,y) \tag{3.4}$$

(which is an improper integral).

We thus obtain the Corollary.

(3.4) <u>Corollary</u>

When M has a pole at $y \in M$, then

$$p^M_t(x,y) = q_t(x,y)[1 + \sum_{j=1}^{N} b_j(x,y)t^j + R'_{N+1}(t,x,y)t^{N+1}]$$

where $b_n(x,y)$, $1 \le n \le N$ and $R'_{N+1}(t,x,y)$ are defined as in Corollary (3.3) above.

<u>Proof.</u>

Immediate by taking $\mathrm{Cut}(y) = \emptyset$ in Corollary (3.3).

(3.5) <u>Remarks</u>

(i) Corollary (3.4) above is just Theorem (3.5) in [37] without the "parasitic" boundedness conditions.

(ii) When both $\frac{LC_y}{C_y}$ and $f_{r_1,\dots,r_n}$; $1 \le n \le N+1$ are bounded, then the improper integrals become proper integrals i.e.

$$R'_{N+1}(t,x,y) = R_{N+1}(t,x,y)$$

$$= \int_0^1 \int_0^{r_1} \cdots \int_0^{r_N} E_x(\chi_{\zeta > tr_{N+1}} f_{r_1,\ldots,r_{N+1}}(x^t(tr_{N+1}))\exp\{\int_0^{tr_{N+1}} \frac{LC_y}{C_y}(x^t(s))ds\})$$

where $\zeta = \zeta(t)$ is the first hitting time of the cut-locus in case $\mathrm{Cut}(y) \neq \emptyset$ and $\zeta = +\infty$ in the case that y is a pole for M.

(iii) These results can be extended to the case that $\exp_y : T_yM \to M$ is a covering map (and not necessarily a diffeomorphism) as in [14] as follows: (See also [16] for details of proof)

$$p_t^M(x,y) = \sum_i p_t^{\tilde{M}}(x_i,\tilde{y})$$

where $(x_i)_i$ are the points of $\exp_y^{-1}(x)$ and $\tilde{y} = 0$ is a pole for $\tilde{M} = T_yM$. In particular if M has non-positive curvature, then $\exp_y : T_yM \to M$ has no conjugate points for all $y \in M$ and hence is a covering map for all $y \in M$.

§4. SMALL TIME ASYMPTOTICS OF THE HEAT KERNEL

The small time asymptotic behaviour of the Heat Kernel has been studied by various authors e.g. Arede [2], Azencott [5], Azencott et al. [6], Bismut [9], Ikeda [23], Kannai [25], Kifer [27], Kusuoka and Stroock [28], McKean and Singer [30], Molchanov [31], Pinsky [34], Taniguchi [36], Watanabe [40], Watling [37].

Here we will first give an asymptotic expansion for the Dirichlet Heat Kernel of any set of compact closure and smooth boundary star-shaped from $y \in M$. We will then extend this result to more general domains and spaces.

(4.1) Theorem (Small Time Asymptotic Expansion Formula in U)

$$p_t^U(x,y) = q_t(x,y)[1+b_1(x,y)t+b_2(x,y)t^2+\dots+b_N(x,y)t^N + o(t^N)]$$

for all positive integers $N \geq 1$ and where for $1 \leq n \leq N$, b_n is given as in Theorem (3.2).

Proof.

Consider $R_{N+1}(t,x,y)$ as given in Theorem (3.2). Both $\frac{LC_y}{C_y}$ and $f_{r_1,\dots,r_{N+1}}$ are bounded in U and hence $R_{N+1}(t,x,y)$ is bounded by a constant $Ce^{k+r_{N+1}}$ for some constants C and k and so clearly,

$$R_{N+1}(t,x,y)t^{N+1} = o(t^N).$$

(4.2) Theorem (Small Time Asymptotic Expansion Formula in M for near points)

$p_t^M(x,y)$ has the same small time asymptotic expansion as $p_t^U(x,y)$ for all $x \in U \subset M\setminus Cut(y)$.

Proof.

By Varadhan's estimate (see for example the proof of Theorem (2.2) in [31]) we have:

$$p_t^M(x,y) \leq p_t^U(x,y) + e^{-\frac{d(x,y)^2+\delta}{2t}} \quad \text{for some } \delta > 0$$

and for all $x \in U$. Consequently,

$$0 \leq \frac{p_t^M(x,y) - p_t^U(x,y)}{q_t(x,y)} \leq \frac{(2\pi t)^{\frac{n}{2}} e^{-\frac{\delta}{2t}}}{C_y(x)} = o(t^N). \qquad (4.1)$$

Hence,

$$p_t^M(x,y) = p_t^U(x,y) + q_t(x,y)\, o(t^N) \tag{4.2}$$

$$= q_t(x,y)[1 + b_1(x,y)t + b_2(x,y)t^2 + \ldots + b_N(x,y)t^N + o(t^N)] \tag{4.3}$$

for all integers $N \geq 1$.

(4.3) Theorem (Small Time Asymptotic Expansion Formula in M for fairly distant points)

For each $x \in M\setminus Cut(y)$, we have:

$$p_t^M(x,y) = q_t(x,y)[1 + b_1(x,y)t + b_2(x,y)t^2 + \ldots + b_N(x,y)t^N + o(t^N)].$$

Proof.

Since $M\setminus Cut(y) = \bigcup_{k\geq 1} U_k$ where each U_k is star-shaped from y, has compact closure and smooth boundary, there exists $k_o \geq 1$ (which we fix) such that $x \in U_{k_o}$ and so by the proof of Theorem (4.2),

$$p_t^M(x,y) = p_t^{U_{k_o}}(x,y) + q_t(x,y)\, o(t^N) \tag{4.4}$$

$$= q_t(x,y)[1 + b_1(x,y)t + b_2(x,y)t^2 + \ldots + b_N(x,y)t^N + o(t^N)]. \tag{4.5}$$

(4.4) Corollary

We also have for each $x \in M\setminus Cut(y)$,

$$p_t^{M\setminus Cut(y)}(x,y) = q_t(x,y)[1 + \sum_{j=1}^{N} b_j(x,y)t^j + o(t^N)].$$

Proof.

Clearly, we have:

$$0 \le p_t^{M\setminus Cut(y)}(x,y) \le p_t^M(x,y) = q_t(x,y)[1 + \sum_{j=1}^{N} b_j(x,y)t^j + o(t^N)] \qquad (4.6)$$

and hence

$$p_t^{M\setminus Cut(y)}(x,y) = q_t(x,y)[1 + \sum_{j=1}^{N} b_j(x,y)t^j + o(t^N)] \qquad (4.7)$$

(4.5) Corollary

When y is a pole for M, we have for all $x \in M$,

$$p_t^M(x,y) = q_t(x,y)[1 + b_1(x,y)t + b_2(x,y)t^2 + \dots + b_N(x,y)t^N + o(t^N)].$$

Proof.

The proof is immediate by taking $Cut(y) = \emptyset$ in Corollary (4.4) above.

(4.6) Remark

The case of a manifold with boundary star-shaped from a point y is treated exactly as $M\setminus Cut(y)$ with boundary ∂M replacing $Cut(y)$ and the exit time from $M\setminus Cut(y)$ becomes the hitting time of the boundary ∂M.

§5. SOME APPLICATIONS OF THE EXPANSION FORMULAE

We first make a combined use of the asymptotic expansions in Chapter III and Chapter IV: We suppose that N is a complete k-dimensional submanifold of the n-dimensional manifold M (as we did in Chapter III). We next suppose that $R_{q+1}(t,x,N)$ (see Theorem (2.4) of Chapter III) is bounded and that $R_{q+1}(t,x,y)$ (see Theorem (3.2) of Chapter IV) is bounded for each $y \in N$. Then we have the following:

(5.1) Proposition

If each $y \in N$ is a pole for M (in particular this is so when M has negative curvature and is simply connected), then

(i) $q_t(x,N) = \int_N f(y)q_t(x,y)dy$

(ii) $b_j(x,N) = \dfrac{\int_N f(y)q_t(x,y)b_j(x,y)dy}{\int_N f(y)q_t(x,y)dy}$ for $1 \le j \le q$.

Proof.

Look at the expansion of $p_t^M(x,y)$:

$$p_t^M(x,y) = q_t(x,y)[1 + \sum_{j=1}^{q} b_j(x,y)t^j + o(t^q)] \tag{5.1}$$

for each $y \in N$, and hence:

$$\int_N f(y)p_t^M(x,y)dy = \int_N q_t(x,y)f(y)[1 + \sum_{j=1}^{q} b_j(x,y)t^j + o(t^q)]dy \tag{5.2}$$

But by Theorem (2.5) of Chapter III,

$$\int_N f(y)p_t^M(x,y)dy = q_t(x,N)[1 + \sum_{j=1}^{q} b_j(x,N)t^j + o(t^q)]. \tag{5.3}$$

Thus comparing coefficients in (5.2) and (5.3) we obtain the result.

Next we combine the use of Milson's Formula and the Exact Expansion Formula of this Chapter to compute the coefficients in the expansion of the

Heat Kernel of the Hyperbolic n-space. In fact Milson's Formula quoted in [11] p. 151 is as follows: Let $L = \frac{1}{2}\Delta$ and let $p_t^n(x,y)$ be the Heat Kernel for H^n relative to L. Then we have:

$$p_t^{n+2}(x,y) = -\frac{e^{-\frac{nt}{2}}}{2\pi \sinh r}\frac{\partial}{\partial r}(p_t^n(x,y)) \tag{5.4}$$

where $r = d(x,y)$.

Suppose that n is odd; then

$$p_t^5(x,y) = -\frac{e^{-\frac{3t}{2}}}{2\pi \sinh r}\frac{\partial}{\partial r}(p_t^3(x,y)). \tag{5.5}$$

Now,

$$\frac{1}{2}\,\theta_y^{\frac{1}{2}}(x)\Delta\theta_y^{-\frac{1}{2}}(x) = -\frac{(n-1)^2}{8} + \frac{(n-1)(n-3)}{8}\left(\frac{1}{r^2} - \frac{1}{\sinh^2 r}\right) \tag{5.6}$$

for H^n. Using the Elementary Heat Kernel Formula given in Corollary (2.6) of this Chapter, we hav e:

$$p_t^3(x,y) = (2\pi t)^{-\frac{3}{2}}\frac{r}{\sinh r}\,e^{-\frac{r^2}{2t}}\,e^{-\frac{t}{2}} \tag{5.7}$$

and hence,

$$p_t^5(x,y) = -\frac{e^{-\frac{3t}{2}}}{2\pi \sinh r}\frac{\partial}{\partial r}(p_t^3(x,y))$$

$$= (2\pi t)^{-\frac{5}{2}}\left(\frac{r}{\sinh r}\right)^2 e^{-\frac{r^2}{2t}}\,e^{-2t}\left[1 + \frac{r\cosh r - \sinh r}{r^2 \sinh r}\cdot t\right] \tag{5.8}$$

Consequently the coefficients $(b_n)1 \leq n \leq N$ and the Remainder term $R_{N+1}(t,x,y)$ in the Exact Expansion of $p_t^3(x,y)$ and $p_t^5(x,y)$ can easily be computed:

For $p_t^3(x,y)$ b_n is just the coefficient of t^n in the expansion of $e^{-\frac{t}{2}}$:

$$b_n = (-1)^n(\tfrac{1}{2})^n \frac{1}{n!} \tag{5.9}$$

and

$$R_{N+1}(t,x,y) = \sum_{n=N+1}^{+\infty} (-1)^n(\tfrac{1}{2})^n \frac{1}{n!} t^n. \tag{5.10}$$

For $p_t^5(x,y)$ b_n is the coefficient of t^n in the expansion of

$$e^{-2t}[1 + \frac{r\cosh r - \sinh r}{r^2 \sinh r} \cdot t]$$

$$= \sum_{n=0}^{\infty} (-2)^n[1 - \frac{n}{2} \frac{r\cosh r - \sinh r}{r^2 \sinh r}] \frac{t^n}{n!}. \tag{5.11}$$

Hence for $1 \leq n \leq N$, and all $N \geq 1$,

$$b_n(x,y) = \frac{(-2)^n}{n!} [1 - \frac{n}{2} \frac{r\cosh r - \sinh r}{r^2 \sinh r}] \tag{5.12}$$

and

$$R_{N+1}(t,x,y) = \lim_{k\uparrow\infty} R_{N+1}^k(t,x,y)$$

(where $R_{N+1}^k(t,x,y)$ is defined in (3.2) of this Chapter) is given by:

$$R_{N+1}(t,x,y) = \sum_{n=N+1}^{+\infty} (-2)^n[1 - \frac{n}{2} \frac{r\cosh r - \sinh r}{r^2 \sinh r}] \frac{t^n}{n!}. \tag{5.13}$$

We can re-apply the formula:

$$p_t^7(x,y) = -\frac{e^{-\frac{5t}{2}}}{2\pi \sinh r}\frac{\partial}{\partial r}(p_t^5(x,y)) \tag{5.14}$$

to compute all the coefficients and Remainder term in the expansion of $p_t^7(x,y)$ and hence for all odd $n \geq 3$.

§6. REPRESENTATION OF $b_1(y,y)$ AND $b_2(y,y)$ in terms of the Curvature at y.

To compute $b_1(y,y)$ and $b_2(y,y)$ (see (4.3) of Theorem (4.2) of this Chapter) in terms of the curvature at y, we will use the power series expansion of the components g_{ij} of the metric tensor in normal coordinates about y and that of the Jacobian determinant θ_y of the exponential map at y. We know that:

$$\theta_y = \sqrt{\det(g_{ij})} \tag{6.1}$$

These expansions are given by Alfred Gray in [21]'.

We start with the computation of $b_1(y,y)$:

$$b_1(y,y) = \int_0^1 (F(1,1-r_1)\frac{LC_y}{C_y})(y)dr_1 \tag{6.2}$$

$$= \int_0^1 \frac{LC_y}{C_y}(\gamma(r_1))dr_1 \tag{6.3}$$

where γ is the unique minimal geodesic from y to y in time 1, i.e.

$$\gamma(s) = y \quad \forall s \in [0,1]$$

and so,

$$\frac{LC_y}{C_y}(\gamma(r_1)) = \frac{LC_y}{C_y}(y). \tag{6.4}$$

Consequently, $b_1(y,y) = \frac{LC_y}{C_y}(y).$ (6.5)

Suppose $V \equiv 0$, then

$$\frac{LC_y}{C_y} = \frac{L(B_y\ \theta_y^{-\frac{1}{2}})}{B_y\ \theta_y^{-\frac{1}{2}}} \tag{6.6}$$

$$= \frac{LB_y}{B_y} + \frac{L\theta_y^{-\frac{1}{2}}}{\theta^{-\frac{1}{2}}} - \frac{1}{2}\langle\nabla \log B_y, \nabla \log\theta_y\rangle \tag{6.7}$$

$$= \frac{1}{2}\ \frac{\Delta B_y}{B_y} + \frac{1}{2}\frac{\Delta\theta_y^{-\frac{1}{2}}}{\theta_y^{-\frac{1}{2}}} + \langle b, \nabla \log B_y\rangle - \frac{1}{2}\langle b, \nabla \log \theta_y\rangle$$

$$- \frac{1}{2}\langle\nabla \log B_y, \nabla \log \theta_y\rangle. \tag{6.8}$$

Now, $B_y(y) = 1 = \theta_y(y)$ and $\nabla \log B_y(y) = - b(y)$ and so,

$$\frac{LC_y}{C_y}(y) = \frac{1}{2}\ \Delta B_y(y) + \frac{1}{2}\ \Delta\theta_y^{-\frac{1}{2}}(y) - \|b(y)\|^2. \tag{6.9}$$

By definition, $\Delta B_y(y) = \text{trace}\ \nabla\nabla\ B_y(y)$ (6.10)

$$= \text{div}\ \nabla B_y(y)$$

$$= \text{div}\ \nabla \log\ B_y(y) + \|\nabla \log\ B_y(y)\|^2$$

$$= - \text{div}\ b(y) + \|b(y)\|^2. \tag{6.11}$$

Hence,

$$\frac{LC_y}{C_y}(y) = \frac{1}{2}\ \Delta\theta_y^{-\frac{1}{2}}(y) - \frac{1}{2}\ \text{div}\ b(y) - \frac{1}{2}\ \|b(y)\|^2 \tag{6.12}$$

We will work in normal coordinates $(x_1, x_2, \ldots, x_d)$ around y:

$$\frac{1}{2}\,\Delta\theta_y^{-\frac{1}{2}}(y) = \frac{1}{2}\,g^{\alpha\beta}(y)\Big(\frac{\partial^2\,\theta_y^{-\frac{1}{2}}}{\partial x_\alpha \partial x_\beta}\,(y) - \Gamma^{\gamma}_{\alpha\beta}(y)\,\frac{\partial\theta_y^{-\frac{1}{2}}}{\partial x_\gamma}(y)\Big) \tag{6.13}$$

$$= \frac{1}{2}\,\frac{\partial^2\,\theta_y^{-\frac{1}{2}}}{\partial x_\alpha^2}(y) \tag{6.14}$$

since $g^{\alpha\beta}(y) = \delta^{\alpha\beta}$ and $\Gamma^{\gamma}_{\alpha\beta}(y) = 0$.

Now,

$$\frac{\partial^2\,\theta_y^{-\frac{1}{2}}}{\partial x_\alpha \partial x_\beta}(y) = \frac{\partial}{\partial x_\alpha}\,\Big(\frac{\partial\theta_y^{-\frac{1}{2}}}{\partial x_\beta}\Big)(y) = -\,\frac{1}{2}\,\frac{\partial}{\partial x_\alpha}\Big(\theta_y^{-3/2}\;\frac{\partial\theta_y}{\partial x_\beta}\Big)(y)$$

$$= -\,\frac{1}{2}\,\Big(\frac{\partial\theta_y^{-3/2}}{\partial x_\alpha}\,(y)\,\frac{\partial\theta_y}{\partial x_\beta}(y) + \theta_y^{-3/2}(y)\,\frac{\partial^2\theta_y}{\partial x_\alpha \partial x_\beta}(y)\Big)$$

$$= -\,\frac{1}{2}\,\frac{\partial^2\theta_y}{\partial x_\alpha \partial x_\beta}\,(y) \tag{6.15}$$

since $\frac{\partial\theta_y}{\partial x_\beta}(y) = 0$ and $\theta_y(y) = 1$.

For the power series expansion of θ_y (see (E_2) given below. From that expansion, we have:

$$\frac{\partial^2\theta_y}{\partial x_\alpha \partial x_\beta}(x) = -\,\frac{1}{6}\,\sum_{i,j=1}^{d} R_{ij}(y)\,\frac{\partial^2}{\partial x_\alpha \partial x_\beta}(x_i x_j)$$

$$-\,\frac{1}{12}\,\sum_{i,j,k=1}^{d}\,\nabla_i R_{jk}(y)\,\frac{\partial^2}{\partial x_\alpha \partial x_\beta}(x_i x_j x_k) + \ldots \tag{6.16}$$

$$= -\,\frac{1}{6}\,\sum_{i,j=1}^{d} R_{ij}(y)\,\frac{\partial}{\partial x_\alpha}\,(x_j\,\delta_{i\beta} + x_i\,\delta_{j\beta}) + O(|x|) \tag{6.17}$$

$$= -\,\frac{1}{6}\,\sum_{i,j=1}^{d} R_{ij}(y)(\delta_{j\alpha}\delta_{i\beta} + \delta_{i\alpha}\delta_{j\beta}) + O(|x|) \tag{6.18}$$

(where $R_{ij}(x) = \sum_{m=1}^{d} R_{imjm}(x)$ are the components of the Ricci Curvature tensor at $x \in M$ and O is Landau's big O).

$$= -\frac{1}{6}(R_{\beta\alpha}(y) + R_{\alpha\beta}(y) + O(|x|). \tag{6.19}$$

Hence, $\dfrac{\partial^2 \theta_y}{\partial x_\alpha \partial x_\beta}(y) = -\dfrac{1}{6}(R_{\beta\alpha}(y) + R_{\alpha\beta}(y))$

$$= -\frac{1}{3} R_{\alpha\beta}(y) \tag{6.20}$$

since $R_{\beta\alpha} = R_{\alpha\beta}$.

Finally by (6.14) and (6.15),

$$\frac{1}{2}\Delta\theta_y^{-\frac{1}{2}}(y) = -\frac{1}{4}\frac{\partial^2\theta_y}{\partial x_\alpha^2}(y) \tag{6.21}$$

$$= (-\tfrac{1}{4})(-\tfrac{1}{3})R_{\alpha\alpha}(y) \text{ by (6.20)}$$

$$= \frac{1}{12} R_{\alpha\alpha}(y) \tag{6.22}$$

i.e. $$\frac{1}{2}\Delta\theta_y^{-\frac{1}{2}}(y) = \frac{1}{12}\sum_{\alpha=1}^{d} R_{\alpha\alpha}(y) \tag{6.23}$$

$$= \frac{\tau(y)}{12} \tag{6.24}$$

where $\tau(x) = \sum_{\alpha=1}^{d} R_{\alpha\alpha}(x) = \sum_{\alpha,m=1}^{d} R_{\alpha m\alpha m}(x)$

is the Scalar Curvature of M at $x \in M$ (we are taking d to be the dimension of M). Consequently,

$$b_1(y,y) = \frac{1}{2}\Delta\theta_y^{-\frac{1}{2}}(y) - \frac{1}{2}\operatorname{div} b(y) - \frac{1}{2}\|b(y)\|^2 \tag{6.25}$$

$$= \frac{\tau(y)}{12} - \frac{1}{2}\,\mathrm{div}\, b(y) - \frac{1}{2}\|b(y)\|^2 . \tag{6.26}$$

We next compute $b_2(y,y)$ in terms of the curvature at $y \in M$. We first obtain the "raw" expression for $b_2(y,y)$:

$$b_2(y,y) = \int_0^1 \int_0^{r_1} F(1,1-r_2)L_{C_y} F(1-r_2,1-r_1)\frac{LC_y}{C_y}(y)dr_1dr_2. \tag{6.27}$$

Set

$$g_{r_1,r_2} = F(1-1-r_2)L_{C_y} F(1-r_2,1-r_1)\frac{LC_y}{C_y} \tag{6.28}$$

and so

$$g_{r_1,r_2}(y) = L_{C_y} F(1-r_2,1-r_1)\frac{LC_y}{C_y}(\gamma(r_2))$$

where γ_2 is the unique minimal geodesic from y to y in time 1; $\gamma_2(s) = y \ \ \forall s \in [0,1]$, and so

$$g_{r_1,r_2}(y) = L_{C_y} F(1-r_2,1-r_1)\frac{LC_y}{C_y}(y)$$

$$= L(C_y F(1-r_2,1-r_1)\frac{LC_y}{C_y})(y) \tag{6.29}$$

since $C_y(y) = 1$.

Thus

$$g_{r_1,r_2}(y) = L(C_y F(1-r_2,1-r_1)\frac{LC_y}{C_y})(y) \tag{6.30}$$

$$= F(1-r_2,1-r_1)\frac{LC_y}{C_y}(y)\cdot LC_y(y) + C_y(y)L(F(1-r_2,1-r_1)\frac{LC_y}{C_y})(y)$$

$$+ \langle\nabla C_y(y),\nabla(F(1-r_2,1-r_1)\frac{LC_y}{C_y})(y)\rangle \tag{6.31}$$

where ∇ now refers to the gradient operator in the normal coordinate system.

Now, $(F(1-r_2,1-r_1)\frac{LC_y}{C_y})(x) = \frac{LC_y}{C_y}(\gamma(r_1-r_2))$ (6.32)

where $\gamma(s) = x - \frac{s}{1-r_2}x$ in normal coordinates and so, $\gamma(r_1-r_2) = \frac{1-r_1}{1-r_2}x$.

Hence

$$(F(1-r_2,1-r_1)\frac{LC_y}{C_y}(y) = \frac{LC_y}{C_y}(y) = LC_y(y)$$

since $C_y(y) = 1$. Consequently,

$$F(1-r_2,1-r_1)\frac{LC_y}{C_y}(y).LC_y(y) = (LC_y(y))^2 \tag{6.33}$$

Next,

$$\frac{\partial}{\partial x_j}(F(1-r_2,1-r_1)\frac{LC_y}{C_y}(x) = \frac{\partial}{\partial x_j}(\frac{LC_y}{C_y} \circ \gamma(r_1-r_2)) \tag{6.34}$$

$$= \frac{\partial}{\partial x_j}\frac{LC_y}{C_y}(\gamma(r_1-r_2)) \circ \frac{\partial}{\partial x_j}\gamma(r_1-r_2) \tag{6.35}$$

$$= F(1-r_2,1-r_1)\frac{\partial}{\partial x_j}\cdot\frac{LC_y}{C_y}(x)\cdot\frac{1-r_1}{1-r_2} \tag{6.36}$$

and

$$\frac{\partial^2}{\partial x_i \partial x_j}(F(1-r_2,1-r_1)\frac{LC_y}{C_y}(x) = \frac{\partial}{\partial x_i}(\frac{\partial}{\partial x_i}F(1-r_2,1-r_1)\frac{LC_y}{C_y})(x) \tag{6.37}$$

$$= \frac{\partial}{\partial x_i}(F(1-r_2,1-r_1)\frac{\partial}{\partial x_j}\frac{LC_y}{C_y})(x)\cdot\frac{1-r_1}{1-r_2} \tag{6.38}$$

$$= F(1-r_2,1-r_1)\frac{\partial^2}{\partial x_i \partial x_j}\frac{LC_y}{C_y}(x)(\frac{1-r_1}{1-r_2})^2. \tag{6.39}$$

Thus,

$$\int_0^1\int_0^{r_1} \frac{\partial}{\partial x_j}(F(1-r_2,1-r_1)\frac{LC_y}{C_y})(y)dr_1dr_2$$

$$= \int_0^1\int_0^{r_1} F(1-r_2,1-r_1)\ \frac{\partial}{\partial x_j}\ \frac{LC_y}{C_y}(y)\cdot\frac{1-r_1}{1-r_2}\ dr_1dr_2 \tag{6.40}$$

$$= \int_0^1\int_0^{r_1} \frac{\partial}{\partial x_j}\ \frac{LC_y}{C_y}(y)\ \frac{1-r_1}{1-r_2}\ dr_1dr_2 \tag{6.41}$$

$$= \frac{\partial}{\partial x_j}\ \frac{LC_y}{C_y}(y)\int_0^1\int_0^{r_1} \frac{1-r_1}{1-r_2}\ dr_1dr_2 = \frac{1}{4}\ \frac{\partial}{\partial x_j}\ \frac{LC_y}{C_y}(y). \tag{6.42}$$

Similarry,

$$\int_0^1\int_0^{r_1} \frac{\partial^2}{\partial x_i\partial x_j}\ (F(1-r_2,1-r_1)\ \frac{LC_y}{C_y})(y)dr_1dr_2$$

$$= \frac{\partial^2}{\partial x_i\partial x_j}\ \frac{LC_y}{C_y}(y)\int_0^1\int_0^{r_1} (\frac{1-r_1}{1-r_2})^2\ dr_1dr_2 \tag{6.43}$$

$$= \frac{1}{6}\ \frac{\partial^2}{\partial x_i\partial x_j}\ \frac{LC_y}{C_y}(y). \tag{6.44}$$

Consequently,

$$b_2(y,y) = \int_0^1\int_0^{r_1} g_{r_1,r_2}(y)dr_1dr_2$$

$$= \frac{1}{2}\ (LC_y(y))^2 + \int_0^1\int_0^{r_1} L(F(1-r_2,1-r_1)\frac{LC_y}{C_y})(y)dr_1dr_2$$

$$+ \langle\nabla C_y(y),\ \int_0^1\int_0^{r_1} \nabla(F(1-r_2,1-r_1)\frac{LC_y}{C_y})(y)dr_1dr_2\rangle \tag{6.45}$$

$$= \frac{1}{2}(LC_y(y))^2 + \frac{1}{2}\cdot\frac{1}{6}\sum_{\alpha,\beta=1}^{d} g^{\alpha\beta}(y)\ \frac{\partial^2}{\partial x_\alpha\partial x_\beta}\cdot\frac{LC_y}{C_y}(y)$$

$$- \frac{1}{2}\cdot\frac{1}{4}\sum_{\alpha,\beta,\gamma=1}^{d} g^{\alpha\beta}(y)\ \Gamma^{\gamma}_{\alpha\beta}(y)\ \frac{\partial}{\partial x_j}\cdot\frac{LC_y}{C_y}(y) \tag{6.46}$$

$$+ \frac{1}{4} \langle b(y), \nabla \frac{LC_y}{C_y}(y)\rangle + \langle \nabla C_y(y), \frac{1}{4} \nabla \frac{LC_y}{C_y}(y)\rangle$$

$$= \frac{1}{2}(LC_y(y))^2 + \frac{1}{12} \sum_{\alpha=1}^{d} \frac{\partial^2}{\partial x_\alpha^2} \cdot \frac{LC_y}{C_y}(y) + \frac{1}{4} \langle b(y) + \nabla C_y(y), \nabla \frac{LC_y}{C_y}(y)\rangle \tag{6.47}$$

since $g^{\alpha\beta}(y) = \delta^{\alpha\beta}$ and $\Gamma^{\gamma}_{\alpha\beta}(y) = 0$.

(6.47) above is the "raw" expression for $b_2(y,y)$.

We shall proceed to compute the above expression in terms of the curvature at y but before we do so, we first give the basic power series expansions necessary for the computations: Let $(x_1,\ldots,x_d)$ be the normal coordinate system around y. Then we have:

(E_1) (given in [21]'; Corollary (2.9)): For x in the neighbourhood of y, we have:

$$g_{pq}(x) = \delta_{pq} - \frac{1}{2} \sum_{i,j=1}^{d} R_{ipjq}(y)x_i x_j - \frac{1}{6} \sum_{i,j,k=1}^{d} \nabla_i R_{jpkq}(y)x_i x_j x_k$$

$$+ \frac{1}{120} \sum_{i,j,k,\ell=1}^{d} \{-6\nabla^2_{ij} R_{kp\ q}(y) + \frac{16}{3} \Sigma R_{ipjs}(y)R_{kq\ell s}(y)\}x_i x_j x_k x_\ell$$

$$+ \frac{1}{90} \sum_{i,j,k,\ell,k=1}^{d} \{ -\nabla^3_{ijk} R_{\ell phq}(y) + 2 \sum_{s=1}^{d} (\nabla_i R_{jpks}(y)R_{\ell qhs}(y)$$

$$\nabla_i R_{jqks}(y)\ R_{\ell phs}(y))\}x_i x_j x_k x_\ell x_h + \cdots$$

where R_{pqrs} are the components of the Riemannian Curvature tensor.

(E_2) (given in [21]'; Corollary (2.10)): For x in the neighbourhood of y, we have:

$$\theta_y(x) = 1 - \frac{1}{6}\sum_{i,j=1}^{d} R_{ij}(y)x_i x_j - \frac{1}{12}\sum_{i,j,k=1}^{d} \nabla_i R_{jk}(y)x_i x_j x_k$$

$$+ \frac{1}{24}\sum_{i,j,k,\ell=1}^{d} \{-\frac{3}{5}\nabla^2_{ij} R_{k\ell}(y) + \frac{1}{3}R_{ij}(y)R_{k\ell}(y) - \frac{2}{15}\sum_{a,b=1}^{d} R_{iajb}(y)R_{ka\ell b}(y)\}x_i x_j x_k x_\ell$$

$$+ \frac{1}{120}\sum_{i,j,k,\ ,h=1}^{d} \{-\frac{2}{3}\nabla^3_{ijk}R_{\ell h}(y) + \frac{5}{3}\nabla_i R_{jk}(y)R_{\ell h}(y)$$

$$- \frac{2}{3}\sum_{a,b=1}^{d} \nabla_i R_{jakb}(y)R_{\ell ahb}(y)\}\ x_i x_j x_k x_\ell x_h$$

$$+ \frac{1}{720}\sum_{i,j,k,\ell,h,g=1}^{d} \{-\frac{5}{7}\nabla^4_{ijk\ell}R_{hg}(y) + 3\nabla^2_{ij}R_{k\ell}(y)R_{hg}(y)$$

$$+ \frac{5}{2}\nabla_i R_{jk}(y)\ \nabla_\ell R_{hg}(y) - \frac{8}{7}\sum_{a,b=1}^{d} \nabla^2_{ij} R_{ka\ell b}(y)R_{hagb}(y)$$

$$- \frac{5}{9}R_{ij}(y)R_{k\ell}(y)R_{hg}(y) - \frac{15}{14}\sum_{a,b=1}^{d} \nabla_i R_{jakb}(y)\ \nabla_\ell R_{hagb}(y)\ .$$

$$- \frac{16}{63}\sum_{a,b,c=1}^{d} R_{iajb}(y)R_{kb\ell c}(y)R_{hcga}(y) + \frac{2}{3}R_{ij}(y)\sum_{a,b=1}^{d}$$

$$R_{ka\ell b}(y)\ R_{bagb}(y)\}x_i x_j x_k x_\ell x_h x_j + \cdots$$

where $R_{ij}(x) = \sum_{k=1}^{d} R_{ikjk}(x)$ are the components of the Ricci Curvature tensor.

The expansion of g^{qr} is not explicitly given in [21]' but we can get one by using the fact that:

$$\sum_{q=1}^{d} g_{pq}g^{qr} = \delta_p^r\ . \tag{6.48}$$

We set $\sigma = \sqrt{x_1^2 + x_2^2 + \ldots + x_d^2}$

and

$$a_i = \frac{x_i}{\sigma} .$$

Suppose that for x in the neighbourhood of y, $g^{qr}(x)$ has an expansion:

$$g^{qr}(x) = \delta^{qr} + \sum_{i,j=1}^{d} R_1(i,j,q,r)x_i x_j + \sum_{i,j,k=1}^{d} R_2(i,j,k,q,r)x_i x_j x_k$$

$$+ \sum_{i,j,k,\ell=1}^{d} R_3(i,j,k,\ell,q,r)x_i x_j x_k x_\ell + \ldots . \tag{6.49}$$

$$= \delta^{qr} + \sum_{i,j=1}^{d} R_s(i,j,q,r)a_i a_j \sigma^2 + \sum_{i,j,k=1}^{d} R_2(i,j,k,q,r)a_i a_j a_k \sigma^3$$

$$+ \sum_{i,j,k,\ =1}^{d} R_3(i,j,k,\ell,q,r)a_i a_j a_k a_\ell\ \sigma^4 + \ldots \tag{6.50}$$

Then we have:

$$\sum_{q=1}^{d} g_{pq}(x)g^{qr}(x) = \sum_{q=1}^{d} [\{\delta_{pq} - \frac{1}{3} \sum_{i,j=1}^{d} R_{ipjq}(y)a_i a_j \sigma^2$$

$$- \frac{1}{6} \sum_{i,j,k=1}^{d} \nabla_i R_{jpkq}(y)a_i a_j a_k \sigma^3 + \frac{1}{120} \sum_{i,j,k,\ell=1}^{d} (-6\nabla_{ij}^2\ R_{kp\ell q}(y)$$

$$+ \frac{16}{3} \sum_{s=1}^{d} R_{ipjs}(y)R_{kq\ell s}(y)a_i a_j a_k a_\ell \sigma^4 + \ldots\}$$

$$\{\delta^{qr} + \sum_{i,j=1}^{d} R_1(i,j,q,r)a_i a_j \sigma^2 + \sum_{i,j,k=1}^{d} R_2(i,j,k,q,r)a_i a_j a_k \sigma^3$$

$$+ \sum_{i,j,k,\ell=1}^{d} R_3(i,j,k,\ell,q,r)a_i a_j a_k a_\ell \sigma^4 + \ldots\ \}]$$

It is clear that the L.H.S. of the above equality is equal to δ_p^r.

Expanding the R.H.S. and equating coefficients of powers of σ, we get:

(1) $$\sum_{i,j,q=1}^{d} \delta_{pq} R_1(i,j,q,r) a_i a_j = \frac{1}{3} \sum_{i,j,q=1}^{d} \delta^{qr} R_{ipjq}(y) a_i a_j$$

i.e. $$\sum_{i,j=1}^{d} R_1(i,j,p,r) a_i a_j = \frac{1}{3} \sum_{i,j=1}^{d} R_{ipjr}(y) a_i a_j.$$

(2) $$\sum_{i,j,k,q=1}^{d} \delta_{pq} R_2(i,j,k,q,r) a_i a_j a_k = \frac{1}{6} \sum_{i,j,k,q=1}^{d} \delta^{qr} \nabla_i R_{jpkq}(y) a_i a_j a_k$$

i.e. $$\sum_{i,j,k=1}^{d} R_2(i,j,k,p,r) a_i a_j a_k = \frac{1}{6} \sum_{i,j,k=1}^{d} \nabla_i R_{jpkr}(y) a_i a_j a_k$$

(3) $$\sum_{i,j,k,\ell=1}^{d} R_3(i,j,k,\ell,p,r) a_i a_j a_k a_\ell$$

$$= \frac{1}{9} \sum_{s=1}^{d} \Big(\sum_{i,j=1}^{d} R_{ipjs}(y) a_i a_j \Big) \Big(\sum_{i,j=1}^{d} R_{isjr}(y) a_i a_j \Big)$$

$$- \frac{1}{120} \sum_{i,j,k,\ell=1}^{d} \Big(-6 \nabla_{ij}^2 R_{kp\ell r}(y) + \frac{16}{3} \sum_{s=1}^{d} R_{ipjs}(y) R_{kr\ell s}(y) \Big) a_i a_j a_k a_\ell$$

Finally using (1), (2) and (3) and replacing p by q, we have:

(E_3): For x in the neighbourhood of y,

$$g^{qr}(x) = \delta^{qr} + \frac{1}{3} \sum_{i,j=1}^{d} R_{iqjr}(y) x_i x_j + \frac{1}{6} \sum_{i,j,k=1}^{d} \nabla_i R_{jqkr}(y) x_i x_j x_k$$

$$- \frac{1}{120} \sum_{i,j,k,\ell=1}^{d} \Big(-6 \nabla_{ij}^2 R_{kq\ell r}(y) + \frac{16}{3} \sum_{s=1}^{d} R_{iqjs}(y) R_{kr\ell s}(y) \Big) x_i x_j x_k x_\ell$$

$$+ \frac{1}{9} \sum_{s=1}^{d} \Big(\sum_{i,j=1}^{d} R_{iqjs}(y) x_i x_j \Big) \Big(\sum_{i,j=1}^{d} R_{isjr}(y) x_i x_j \Big) + \ldots$$

We will assume, just for simplicity, that $b \equiv 0$; hence,

$$b_2(y,y) = \frac{1}{2}\left(\frac{1}{2}\Delta\theta_y^{-\frac{1}{2}}(y)\right)^2 + \frac{1}{24}\sum_{\alpha=1}^{d}\frac{\partial^2}{\partial x_\alpha^2}(\theta_y^{\frac{1}{2}}\Delta\theta_y^{-\frac{1}{2}})(y) \tag{6.51}$$

$$= \frac{1}{2}\left(\frac{\tau(y)}{12}\right)^2 + \frac{1}{24}\sum_{\alpha=1}^{d}\frac{\partial^2}{\partial x_\alpha^2}(\theta_y^{\frac{1}{2}}\Delta\theta_y^{-\frac{1}{2}})(y) \tag{6.52}$$

Now,

$$\frac{\partial^2}{\partial x_\alpha^2}(\theta_y^{\frac{1}{2}}\Delta\theta_y^{-\frac{1}{2}})(y)$$

$$= \frac{\partial^2\theta_y^{\frac{1}{2}}}{\partial x_\alpha^2}(y)\Delta\theta_y^{-\frac{1}{2}}(y) + \theta_y^{\frac{1}{2}}(y)\frac{\partial^2}{\partial x_\alpha^2}(\Delta\theta_y^{-\frac{1}{2}})(y)$$

$$+ 2\,\frac{\partial\theta_y}{\partial x_\alpha}(y)\cdot\frac{\partial}{\partial x_\alpha}(\Delta\theta_y^{-\frac{1}{2}})(y) \tag{6.53}$$

$$= \frac{\partial^2\theta_y^{\frac{1}{2}}}{\partial x_\alpha^2}(y)\Delta\theta_y^{-\frac{1}{2}}(y) + \frac{\partial^2}{\partial x_\alpha^2}(\Delta\theta_y^{-\frac{1}{2}})(y) \tag{6.54}$$

since $\theta_y(y) = 1$ and $\frac{\partial\theta_y^{\frac{1}{2}}}{\partial x_\alpha}(y) = 0$

Now,

$$\frac{\partial^2\theta_y^{\frac{1}{2}}}{\partial x_\alpha^2}(y) = \frac{1}{2}\,\frac{\partial^2\theta_y}{\partial x_\alpha^2}(y) \tag{6.55}$$

and by the expansion given in (E_2),

$$\frac{1}{2}\,\frac{\partial^2\theta_y}{\partial x_\alpha^2}(x) = -\frac{1}{12}\sum_{i,j=1}^{d}R_{ij}(y)\frac{\partial^2}{\partial x_\alpha^2}(x_ix_j) + O(|x|) \tag{6.56}$$

$$= -\frac{1}{12}\sum_{i,j=1}^{d}R_{ij}(y)(\delta_{j\alpha}\delta_{i\alpha} + \delta_{i\alpha}\delta_{j\alpha}) + O(|x|) \tag{6.57}$$

$$= -\frac{1}{6}\sum_{i,j=1}^{d} R_{ij}(y)\delta_{i\alpha}\delta_{j\alpha} + O(|x|). \qquad (6.58)$$

Hence

$$\frac{1}{2}\frac{\partial^2\theta_y}{\partial x_\alpha^2}(y) = -\frac{1}{6}R_{\alpha\alpha}(y) \qquad (6.59)$$

$$= -\frac{\tau(y)}{6} \qquad (6.60)$$

by summing over α. On the other hand we already know that

$$\Delta\theta_y^{-\frac{1}{2}}(y) = \frac{\tau(y)}{6} \qquad (6.61)$$

and so

$$\frac{\partial^2\theta_y^{\frac{1}{2}}}{\partial x_\alpha^2}(y)\Delta\theta_y^{-\frac{1}{2}}(y) = -\frac{\tau^2(y)}{36}. \qquad (6.62)$$

Consequently,

$$b_2(y,y) = \frac{\tau^2(y)}{288} + \frac{1}{24}\left[-\frac{{}^2(y)}{36} + \sum_{\alpha=1}^{d}\frac{\partial^2}{\partial x^2}(\Delta\theta_y^{-\frac{1}{2}})(y)\right] \qquad (6.63)$$

$$= \frac{\tau^2(y)}{432} + \frac{1}{24}\sum_{\alpha=1}^{d}\frac{\partial^2}{\partial x_\alpha^2}(\Delta\theta_y^{-\frac{1}{2}})(y). \qquad (6.64)$$

Next consider,

$$\frac{\partial^2}{\partial x_\alpha^2}(\Delta\theta_y^{-\frac{1}{2}})(y) = \frac{\partial^2}{\partial x_\alpha^2}\left[\sum_{q,r=1}^{d} g^{qr}\left(\frac{\partial^2\theta_y^{-\frac{1}{2}}}{\partial x_q\partial x_r} - \Gamma_{qr}^{s}\frac{\partial\theta_y^{-\frac{1}{2}}}{\partial x_s}\right)\right](y) \qquad (6.65)$$

$$= \sum_{q,r=1}^{d}\frac{\partial^2}{\partial x_\alpha^2}\left(g^{qr}\frac{\partial^2\theta_y^{-\frac{1}{2}}}{\partial x_q\partial x_r}\right)(y) - \sum_{q,r,s=1}^{d}\frac{\partial^2}{\partial x^2}\left(g^{qr}\Gamma_{qr}^{s}\frac{\partial\theta^{-\frac{1}{2}}}{\partial x_s}\right)(y) \qquad (6.66)$$

$$= \sum_{q,r=1}^{d} \{ \frac{\partial^2 g^{qr}}{\partial x_\alpha^2}(y) \frac{\partial^2 \theta_y^{-\frac{1}{2}}}{\partial x_q \partial x_r}(y) + g^{qr}(y) \frac{\partial^2}{\partial x_\alpha^2} (\frac{\partial^2 \theta_y^{-\frac{1}{2}}}{\partial x_q \partial x_r})(y)$$

$$+ 2 \frac{\partial g^{qr}}{\partial x_\alpha}(y) \frac{\partial}{\partial x_\alpha} (\frac{\partial^2 \theta_y^{-\frac{1}{2}}}{\partial x_q \partial x_r})(y)\} \tag{6.67}$$

$$- \sum_{q,r,s=1}^{d} \{\frac{\partial^2 g^{qr}}{\partial x_\alpha^2}(y) \Gamma_{qr}^s(y) \frac{\partial \theta_y^{-\frac{1}{2}}}{\partial x_s}(y) + g^{qr}(y) \frac{\partial^2}{\partial x_\alpha^2} (\Gamma_{qr}^s \frac{\partial \theta^{-\frac{1}{2}}}{\partial x_s})(y)$$

$$+ 2 \frac{\partial g^{qr}}{\partial x_\alpha}(y) \frac{\partial}{\partial x_\alpha}(\Gamma_{qr}^s \frac{\partial \theta_y^{-\frac{1}{2}}}{\partial x_s})(y)\}$$

$$= \sum_{q,r=1}^{d} \{\frac{\partial^2 g^{qr}}{\partial x_\alpha^2}(y) \frac{\partial^2 \theta_y^{-\frac{1}{2}}}{\partial x_q \partial x_r}(y) + \frac{\partial^2}{\partial x_\alpha^2} (\frac{\partial^2 \theta_y^{-\frac{1}{2}}}{\partial x_q^2})(y)\}$$

$$- \sum_{q,s=1}^{d} \frac{\partial^2}{\partial x_\alpha^2} (\Gamma_{qq}^s \frac{\partial \theta_y^{-\frac{1}{2}}}{\partial x_s})(y) \tag{6.68}$$

(since $g^{qr}(y) = \delta^{qr}$ and $\frac{\partial g^{qr}}{\partial x_\alpha}(y) = 0 = \Gamma_{qr}^s(y)$)

$$= (I_1) + (I_2) - (I_3).$$

Now, the expansion of $g^{qr}(x)$ given by (E_3) is:

$$g^{qr}(x) = \delta^{qr} + \frac{1}{3} \sum_{i,j=1}^{d} R_{iqjr}(y) x_i x_j + \frac{1}{6} \sum_{i,j,k=1}^{d} \{ \quad \} x_i x_j x_k + \cdots$$

Hence

$$\frac{\partial^2 g^{qr}}{\partial x_\alpha^2}(x) = \frac{1}{3} \sum_{i,j=1}^{d} R_{iqjr}(y) \frac{\partial^2}{\partial x_\alpha^2} (x_i x_j) + O(|x|)$$

$$= \frac{1}{3} \sum_{i,j=1}^{d} R_{iqjr}(y)(\delta_{j\alpha}\delta_{i\alpha} + \delta_{i\alpha}\delta_{j\alpha}) + O(|x|) \tag{6.69}$$

and so

$$\frac{\partial^2 g^{qr}}{\partial x_\alpha^2}(y) = \frac{2}{3} R_{\alpha q \alpha r}(y) \tag{6.70}$$

$$\frac{\partial^2 \theta_y^{-\frac{1}{2}}}{\partial x_q \partial x_r}(y) = -\frac{1}{2}\frac{\partial^2 \theta_y}{\partial x_q \partial x_r}(y) \tag{6.71}$$

$$= -\frac{1}{2}[-\frac{1}{6}\Sigma R_{ij}(y)\frac{\partial^2}{\partial x_q \partial x_r}(x_i x_j)] \tag{6.72}$$

$$= \frac{1}{12}\sum_{i,j=1}^{d} R_{ij}(y)(\delta_{jq}\delta_{ir} + \delta_{iq}\delta_{jr}) \tag{6.73}$$

$$= \frac{1}{12}(R_{rq}(y) + R_{qr}(y)) \tag{6.74}$$

$$= \frac{1}{6} R_{qr}(y) \text{ since } R_{rq} = R_{qr} . \tag{6.75}$$

Hence $(I_1) = \frac{2}{3} \cdot \frac{1}{6} R_{\alpha q \alpha r}(y) \cdot R_{qr}(y)$. (6.76)

Consequently,

$$\sum_{\alpha,q,r=1}^{d} \frac{\partial^2 g^{qr}}{\partial x_\alpha^2}(y)\frac{\partial^2 \theta_y^{-\frac{1}{2}}}{\partial x_q \partial x_r}(y) = \frac{1}{9}\sum_{\alpha,q,r=1}^{d} R_{\alpha q \alpha r}(y) R_{qr}(y) \tag{6.77}$$

$$= \frac{1}{9}\sum_{q,r=1}^{d} R_{qr}(y)R_{qr}(y) = \frac{1}{9}\sum_{q,r=1}^{d} R_{qr}^2(y) \tag{6.78}$$

$$= \frac{1}{9}\|\rho(y)\|^2 = (I_1). \tag{6.79}$$

Next consider $(I_3) = \frac{\partial^2}{\partial x_\alpha^2}(\Gamma_{qq}^s \frac{\partial \theta_y^{-\frac{1}{2}}}{\partial x_s})(y)$

$$= \frac{\partial^2 \Gamma_{qq}^s}{\partial x_\alpha^2}(y)\frac{\partial \theta_y^{-\frac{1}{2}}}{\partial x_s}(y) + \Gamma_{qq}^s(y)\frac{\partial^2}{\partial x_\alpha^2}(\frac{\partial \theta_y^{-\frac{1}{2}}}{\partial x_s})(y) \tag{6.80}$$

$$+ 2 \cdot \frac{\partial \Gamma^s_{qq}}{\partial x_\alpha}(y) \cdot \frac{\partial^2 \theta_y^{-\frac{1}{2}}}{\partial x_\alpha \partial x_s}(y)$$

$$= 2 \cdot \frac{\partial \Gamma^s_{qq}}{\partial x_\alpha}(y) \cdot \frac{\partial^2 \theta_y^{-\frac{1}{2}}}{\partial x_\alpha \partial x_s}(y) \qquad (6.81)$$

since

$$\frac{\partial \theta_y^{-\frac{1}{2}}}{\partial x_\alpha}(y) = 0 = \Gamma^s_{qq}(y).$$

Now,

$$\Gamma^s_{qr} = \frac{1}{2} g^{sh}\left(\frac{\partial g_{hr}}{\partial x_q} + \frac{\partial g_{hq}}{\partial x_r} - \frac{\partial g_{qr}}{\partial x_h}\right).$$

Hence

$$\Gamma^s_{qq} = \frac{1}{2} g^{sh}\left(2\frac{\partial g_{hq}}{\partial x_q} - \frac{\partial g_{qq}}{\partial x_h}\right)$$

Thus

$$2\frac{2\Gamma^s_{qq}}{\partial x_\alpha}(y) = \frac{\partial g^{sh}}{\partial x_\alpha}(y)\left(2\frac{\partial g_{hq}}{\partial x_q}(y) - \frac{\partial g_{qq}}{\partial x_h}\right)$$

$$+ g^{sh}(y)\left(2\frac{\partial^2 g_{hq}}{\partial x_\alpha \partial x_q}(y) - \frac{\partial^2 g_{qq}}{\partial x_\alpha \partial x_h}(y)\right)$$

$$= 2\frac{\partial^2 g_{sq}}{\partial x_\alpha \partial x_q}(y) - \frac{\partial^2 g_{qq}}{\partial x_\alpha \partial x_s}(y)$$

since $g^{sh}(y) = \delta^{sh}$ and $\frac{\partial g^{sh}}{\partial x_\alpha}(y) = 0$.

From the expansion of $g_{sq}(x)$ given in (E_1):

$$g_{sq}(x) = \delta_{sq} - \frac{1}{3}\sum_{i,j=1}^{d} R_{isjq}(y)x_i x_j - \frac{1}{6}\sum_{i,j,k=1}^{d} \{\quad\} x_i x_j x_k$$

$$+ \ldots$$

$$\frac{\partial^2 g_{sq}}{\partial x_\alpha \partial x_q}(x) = -\frac{1}{3}\sum_{i,j=1}^{d} R_{isjq}(y)\frac{\partial^2}{\partial x_\alpha \partial x_q}(x_i x_j)$$

$$+ O(|x|)$$

$$= -\frac{1}{3}\sum_{i,j=1}^{d} R_{isjq}(y)(\delta_{j\alpha}\delta_{iq} + \delta_{i\alpha}\delta_{jq}) + O(|x|)$$

$$= -\frac{1}{3}(R_{qs\alpha q}(y) + R_{\alpha sqq}(y)) + O(|x|)$$

$$= -\frac{1}{3}R_{qs\alpha q}(y) + O(|x|)$$

since $R_{\alpha sqq} = 0$.

Hence

$$2\frac{\partial^2 g_{sq}}{\partial x_\alpha \partial x_q}(y) = -\frac{2}{3}R_{qs\alpha q}(y) \qquad (6.82)$$

$$= \frac{2}{3}R_{\alpha qsq}(y). \qquad (6.83)$$

On the other hand,

$$\frac{\partial^2 g_{qq}}{\partial x_\alpha \partial x_s}(x) = -\frac{1}{3}(R_{sq\alpha q}(y) + R_{\alpha qsq}(y)) + O(|x|)$$

$$= -\frac{1}{3}(R_{\alpha qsq}(y) + R_{\alpha qsq}(y)) + O(|x|)$$

$$= -\frac{2}{3}R_{\alpha qsq}(y) + O(|x|)$$

and so $\dfrac{\partial^2 g_{qq}}{\partial x_\alpha \partial x_s}(y) = -\dfrac{2}{3}R_{\alpha qsq}(y)$. $\qquad (6.84)$

Finally,

$$2\,\frac{\partial\Gamma^{s}_{qq}}{\partial x_\alpha}(y) = 2\,\frac{\partial^2 g_{sq}}{\partial x\,\partial x_q}(y) - \frac{\partial^2 g_{qq}}{\partial x\,\partial x_s}(y)$$

$$= \frac{2}{3}R_{\alpha qsq}(y) - \left(-\frac{2}{3}R_{\alpha qsq}(y)\right) = \frac{4}{3}R_{\alpha qsq}(y) \qquad (6.85)$$

$$\frac{\partial^2\,\theta_y^{-\frac{1}{2}}}{\partial x_\alpha \partial x_s}(y) = -\frac{1}{2}\,\frac{\partial^2\theta_y}{\partial x_\alpha \partial x_s}(y) = -\frac{1}{2}\left(-\frac{1}{3}R_{\alpha s}(y)\right)$$

$$= \frac{1}{6}R_{\alpha s}(y). \qquad (6.86)$$

Hence

$$(I_3) = 2\,\frac{\partial\Gamma^{s}_{qq}}{\partial x_\alpha}(y)\,\frac{\partial^2\,\theta_y^{-\frac{1}{2}}}{\partial x_\alpha \partial x_s}(y)$$

$$= \left(\frac{4}{3}R_{\alpha qsq}(y)\right)\left(\frac{1}{6}R_{\alpha s}(y)\right)$$

$$= \frac{2}{9}R_{\alpha qsq}(y)R_{\alpha s}(y)$$

$$= \frac{2}{9}R_{\alpha s}(y)R_{\alpha s}(y)$$

by summing over the index q

$$= \frac{2}{9}\sum_{\alpha,s=1}^{d} R^2_{\alpha s}(y) = \frac{2}{9}\,\|\rho(y)\|^2 \qquad (6.87)$$

by summing over α and s.

We next consider the more difficult case:

$$(I_2) = \frac{\partial^2}{\partial x_\alpha^2}\left(\frac{\partial^2\,\theta_y^{-\frac{1}{2}}}{\partial x_q^2}\right)(y).$$

Now,

$$\frac{\partial^2 \theta_y^{-\frac12}}{\partial x_q^2} = \frac{\partial}{\partial x_q}\left(\frac{\partial \theta_y^{-\frac12}}{\partial x_q}\right)$$

$$= \frac{\partial}{\partial x_q}\left(-\frac{1}{2}\,\theta_y^{-3/2}\cdot\frac{\partial \theta_y}{\partial x_q}\right)$$

$$= -\frac{1}{2}\left[\frac{\partial \theta_y^{-3/2}}{\partial x_q}\cdot\frac{\partial \theta_y}{\partial x_q} + \theta_y^{-3/2}\cdot\frac{\partial^2 \theta_y}{\partial x_q^2}\right].$$

Hence,

$$\frac{\partial^2}{\partial x_\alpha^2}\left[\frac{\partial^2 \theta_y^{-\frac12}}{\partial x_q^2}\right](y)$$

$$= -\frac{1}{2}\left[\frac{\partial^2 \theta_y^{-3/2}}{\partial x_\alpha^2 \partial x_q}(y)\cdot\frac{\partial \theta_y}{\partial x_q}(y) + \frac{\partial \theta_y^{-3/2}}{\partial x_q}(y)\cdot\frac{\partial^3 \theta_y}{\partial x_\alpha^2 \partial x_q}(y)\right.$$

$$+ 2\,\frac{\partial^2 \theta_y^{-3/2}}{\partial x_\alpha \partial x_q}(y)\cdot\frac{\partial^2 \theta_y}{\partial x_\alpha \partial x_q}(y) + \frac{\partial^2 \theta_y^{-3/2}}{\partial x_\alpha^2}(y)\cdot\frac{\partial^2 \theta y}{\partial x_q^2}(y)$$

$$\left. + \theta_y^{-3/2}(y)\cdot\frac{\partial^4 \theta_y}{\partial x_\alpha^2 \partial x_q^2}(y) + 2\cdot\frac{\partial \theta_y^{-3/2}}{\partial x_\alpha}(y)\,\frac{\partial^3 \theta_y}{\partial x_\alpha \partial x_q^2}(y)\right]$$

$$= -\frac{1}{2}\left[2\,\frac{\partial^2 \theta_y^{-3/2}}{\partial x_\alpha \partial x_q}(y)\cdot\frac{\partial^2 \theta y}{\partial x_\alpha \partial x_q}(y) + \frac{\partial^2 \theta_y^{-3/2}}{\partial x_\alpha^2}(y)\,\frac{\partial^2 \theta y}{\partial x_q^2}(y)\right.$$

$$\left. + \frac{\partial^4 \theta_y}{\partial x_\alpha^2 \partial x_q^2}(y)\right]$$

(since $\dfrac{\partial \theta_y^{-3/2}}{\partial x_\alpha}(y) = 0 = \dfrac{\partial \theta_y}{\partial x_q}(y) = \dfrac{\partial \theta_y^{-3/2}}{\partial x_q}(y)$ and $\theta_y^{-3/2}(y) = 1$)

$$= \frac{3}{2}\left(\frac{\partial^2 \theta_y}{\partial x_\alpha \partial x_q}(y)\right)^2 + \frac{3}{4}\,\frac{\partial^2 \theta y}{\partial x_\alpha^2}(y)\,\frac{\partial^2 \theta y}{\partial x_q^2}(y) - \frac{1}{2}\,\frac{\partial^4 \theta_y}{\partial x_\alpha^2 \partial x_q^2}$$

(since $\dfrac{\partial^2 \theta_y^{-3/2}}{\partial x_\alpha \partial x_q}(y) = -\dfrac{3}{2}\,\dfrac{\partial^2 \theta_y}{\partial x_\alpha \partial x_q}(y)$)

$$= \frac{3}{2}(-\frac{1}{3}R_{\alpha q}(y))^2 + \frac{3}{4}(-\frac{1}{3}R_{\alpha\alpha}(y))(-\frac{1}{3}R_{qq}(y)) - \frac{1}{2}\frac{\partial^4\theta_y}{\partial x_\alpha^2 \partial x_q^2}(y)$$

$$= \frac{1}{6}R_{\alpha q}^2(y) + \frac{1}{12}R_{\alpha\alpha}(y)\cdot R_{qq}(y) - \frac{1}{2}\frac{\partial^4\theta_y}{\partial x_\alpha^2 \partial x_q^2}(y)$$

$$= \frac{1}{6}\|\rho(y)\|^2 + \frac{1}{12}\tau^2(y) - \frac{1}{2}\sum_{\alpha,q=1}^{d}\frac{\partial^4\theta_y}{\partial x^2 \partial x_q^2}(y) \qquad (6.88)$$

by summing over α and q.

We finally consider $\frac{\partial^4\theta_y}{\partial x_\alpha^2 \partial x_q^2}(y)$. By the power series expansion of θ_y given by (E_2):

$$\frac{\partial^4\theta_y}{\partial x^2 \partial x_q^2}(y) = A \frac{4}{\partial x^2 \partial x_q^2}(x_i x_j x_k x_\ell) \quad \text{where}$$

$$A = \frac{1}{24}\sum_{i,j,k,\ell=1}^{d}\{-\frac{3}{5}\nabla_{ij}^2 R_{k\ell}(y) + \frac{1}{3}R_{ij}(y)R_{k\ell}(y) - \frac{2}{15}\sum_{a,b=1}^{d}R_{iajb}(y)R_{ka\ell b}(y)\}.$$

Now,

$$\frac{\partial^4}{\partial x_\alpha^2 \partial x_q^2}(x_i x_j x_k x_\ell) = \frac{\partial^2}{\partial x_\alpha^2}\left(\frac{\partial^2}{\partial x_q^2}(x_i x_j x_k x_\ell)\right)$$

$$= \frac{\partial^2}{\partial x_\alpha^2}[x_i x_j \frac{\partial^2}{\partial x_q^2}(x_k x_\ell) + x_k x_\ell \frac{\partial^2}{\partial x_q^2}(x_i x_j) + 2\frac{\partial}{\partial x_q}(x_i x_j)\frac{\partial}{\partial x_q}(x_k x_\ell)]$$

$$= \frac{\partial^2}{\partial x_\alpha^2}[2x_i x_j \delta_{kq}\delta_{\ell q} + 2x_k x_\ell \delta_{iq}\delta_{jq} + 2(x_i\delta_{jq} + x_j\delta_{iq})(x_k\delta_{\ell q} + x_\ell\delta_{kq})]$$

$$= \frac{\partial^2}{\partial x^2}[2x_i x_j\ \delta_{kq}\delta_{\ell q} + 2x_k x_\ell \delta_{iq}\ {}_{jq} + 2\ x_i x_k \delta_{jq}\delta_{\ell q}$$

$$+ 2x_i x_\ell \delta_{jq}\delta_{kq} + 2x_j x_k \delta_{iq}\delta_{\ell q} + 2x_j x_\ell \delta_{iq}\delta_{kq}]$$

$$= 4[\delta_{i\alpha}\delta_{j\alpha}\delta_{kq}\delta_{\ell q} + \delta_{k\alpha}\delta_{\ell\alpha}\delta_{iq}\delta_{jq} + \delta_{i\alpha}\delta_{k\alpha}\delta_{jq}\delta_{\ell q}$$

$$+ \delta_{i\alpha}\delta_{\ell\alpha}\delta_{jq}\delta_{kq} + \delta_{j\alpha}\delta_{k\alpha}\delta_{iq}\delta_{\ell q} + \delta_{j\alpha}\delta_{\ell\alpha}\delta_{iq}\delta_{kq}].$$

Consequently,

$$A. \frac{\partial^4}{\partial x_\alpha^2 \partial x_q^2}(x_i x_j x_k x_\ell)$$

$$= \frac{4}{24}\{- \frac{3}{5}\nabla^2_{\alpha\alpha}R_{qq} + \frac{1}{3}R_{\alpha\alpha}R_{qq} - \frac{2}{15}\sum_{a,b=1}^{d} R_{\alpha a\alpha b}\, R_{qaqb}$$

$$- \frac{3}{5}\Delta^2_{qq}R_{\alpha\alpha} + \frac{1}{3}R_{qq}R_{\alpha\alpha} - \frac{2}{15}\sum_{a,b=1}^{d} R_{qaqb}\, R_{\alpha a\alpha b}$$

$$- \frac{3}{5}\nabla^2_{\alpha q}R_{\alpha q} + \frac{1}{3}R_{\alpha q}R_{\alpha q} - \frac{2}{15}\sum_{a,b=1}^{d} R_{\alpha aqb}\, R_{\alpha aqb}$$

$$- \frac{3}{5}\nabla^2_{\alpha q}R_{q\alpha} + \frac{1}{3}R_{\alpha q}R_{q\alpha} - \frac{2}{15}\sum_{a,b=1}^{d} R_{\alpha aqb}R_{qa\alpha b}$$

$$- \frac{3}{5}\nabla^2_{q\alpha}R_{\alpha q} + \frac{1}{3}R_{q\alpha}R_{\alpha q} - \frac{2}{15}\sum_{a,b=1}^{d} R_{qa\alpha b}R_{\alpha aqb}$$

$$- \frac{3}{5}\nabla^2_{q\alpha}R_{q\alpha} + \frac{1}{3}R_{q\alpha}R_{q\alpha} - \frac{2}{15}\sum_{a,b=1}^{d} R_{qa\alpha b}R_{qa\alpha b}\}(y).$$

Recall that $R_{\alpha q} = R_{q\alpha}$. It is well known (see for example [30] p.65) that

$$\sum_{\alpha,q=1}^{d} \nabla^2_{qq} R_{\alpha\alpha} = \Delta\tau = \sum_{\alpha,q=1}^{d} \nabla^2_{\alpha\alpha} R_{qq} \tag{6.89}$$

and

$$\sum_{\alpha,q=1}^{d} \nabla^2_{\alpha q} R_{\alpha q} = \frac{1}{2}\Delta\tau = \sum_{\alpha,q=1}^{d} \nabla^2_{q\alpha} R_{q\alpha} \tag{6.90}$$

Note:

I am using the sign convention given in T.J. Willmore's book: "Total Curvature in Riemannian Geometry" as opposed to the sign convention adopted by Nomizu & Kobayashi which is the one used by McKean & Singer.

From the foregoing, we see that

$$\sum_{\alpha,q=1}^{d} \frac{\partial^4\theta_y}{\partial x_\alpha^2 \partial x_q^2}(y)$$

$$= \frac{4}{24} \sum_{\alpha,q=1}^{d} \{- \frac{3}{5} \times 4\Delta\tau + \frac{1}{3}(2R_{\alpha\alpha}R_{qq} + 4\, R_{\alpha q}^2) \tag{6.91}$$

$$- \frac{2}{15} \sum_{a,b=1}^{d} (2R_{\alpha a\alpha b}R_{qbqb} + R_{\alpha aqb}^2 + 2R_{\alpha aqb}R_{qa\alpha b} + R_{qa\alpha b}^2)\}(y)$$

Now,

$$\sum_{\alpha,q=1}^{d} R_{\alpha\alpha}R_{qq} = (\sum_{\alpha=1}^{d} R_{\alpha\alpha})(\sum_{q=1}^{d} R_{qq}) = \tau^2$$

$$\sum_{\alpha,q=1}^{d} R_{\alpha q}^2 = \|\rho\|^2$$

$$\sum_{\alpha,a,b,q=1}^{d} R_{\alpha a\alpha b}R_{qaqb} = \sum_{a,b=1}^{d} (\sum_{\alpha=1}^{d} R_{\alpha a\alpha b})(\sum_{q=1}^{d} R_{qaqb})$$

$$= \sum_{a,b=1}^{d} (R_{ab})(R_{ab}) = \sum_{a,b=1}^{d} R_{ab}^2 = \|\rho\|^2$$

$$2R_{\alpha aqb}R_{qa\alpha b} = 2(-R_{\alpha abq})(-R_{qab\alpha})$$

$$= 2R_{\alpha abq}R_{b\alpha qa}$$

$$= 2R_{\alpha abq}R_{\alpha baq}$$

$$= 2.\, \frac{1}{2} R_{\alpha abq}R_{\alpha abq}$$

(by [41] p. 37 No. (1.16))

$$= R^2_{\alpha abq} \text{ and hence } \sum_{\alpha,a,b,q=1}^{d} R^2_{\alpha abq} = \|R\|^2 .$$

Consequently,

$$\frac{\partial^4\theta}{\partial x_\alpha^2 \partial x_q^2}(y)$$

$$= \frac{4}{24}\{-\frac{12}{5}\Delta\tau \ + \frac{1}{3}(2\tau^2 + 4 \ \|\rho\|^2)$$

$$- \frac{2}{15}(2. \ \|\rho\|^2 + \|R\|^2 + \|R\|^2 + \|R\|^2)\}(y)$$

$$= \frac{4}{24}\{-\frac{12}{5}\Delta\tau + \frac{2}{3}\tau^2 + \frac{4}{3}\|\rho\|^2 - \frac{4}{15}\|\rho\|^2 - \frac{6}{15}\|R\|^2\}(y)$$

$$= \frac{4}{24}\{-\frac{12}{5}\Delta\tau + \frac{2}{3}\tau^2 + \frac{16}{15}\|\rho\|^2 - \frac{2}{5}\|R\|^2\}(y)$$

$$= -\frac{2}{5}\Delta\tau(y) + \frac{1}{9}\tau^2(y) + \frac{8}{45}\|\rho(y)\|^2 - \frac{1}{15}\|R(y)\|^2 . \tag{6.92}$$

Hence by (6.88) and (6,92),

$$(I_2) = \frac{1}{6}\|\rho(y)\|^2 + \frac{1}{12}\tau^2(y) - \frac{1}{2}(-\frac{2}{5}\Delta\tau(y) + \frac{1}{9}\tau^2(y)$$

$$+ \frac{8}{45}\|\rho(y)\|^2 - \frac{1}{15}\|R(y)\|^2)$$

$$= \frac{1}{6}\|\rho(y)\|^2 + \frac{1}{12}\tau^2(y) + \frac{1}{5}\Delta\tau(y) - \frac{1}{18}\tau^2(y)$$

$$- \frac{4}{45}\|\rho(y)\|^2 + \frac{1}{30}\|R(y)\|^2$$

$$= \frac{7}{90}\|\rho(y)\|^2 + \frac{1}{36}\tau^2(y) + \frac{1}{5}\Delta\tau(y) + \frac{1}{30}\|R(y)\|^2 .$$

Recall that:

$$b_2(y,y) = \frac{\tau^2(y)}{432} + \frac{1}{24}\,((I_1) - (I_3) + (I_2))$$

$$(I_1) = \frac{1}{9}\,\|\rho(y)\|^2$$

$$(I_3) = \frac{2}{9}\,\|\rho(y)\|^2$$

and so,

$$b_2(y,y) = \frac{\tau^2(y)}{432} + \frac{1}{24}\,\Big(\frac{\|\rho(y)\|^2}{9} - \frac{2\,\|\rho(y)\|^2}{9}$$

$$+ \frac{7\,\|\rho(y)\|^2}{90} + \frac{\tau^2(y)}{36} + \frac{\Delta\tau(y)}{5} + \frac{\|R(y)\|^2}{30}\Big)$$

$$= \frac{\tau^2(y)}{432} + \frac{1}{24}\,\Big(- \frac{\|\rho(y)\|^2}{30} + \frac{\tau^2(y)}{36} + \frac{\Delta\tau(y)}{5} + \frac{\|R(y)\|^2}{30}\Big)$$

$$= \frac{\tau^2(y)}{432} + \frac{\tau^2(y)}{24\times 36} - \frac{\|\rho(y)\|^2}{24\times 30} + \frac{\|R(y)\|^2}{24\times 30} + \frac{\Delta\tau(y)}{24\times 5}$$

$$= \frac{\tau^2(y)}{432} + \frac{\tau^2(y)}{864} - \frac{\|\rho(y)\|^2}{720} + \frac{\|R(y)\|^2}{720} + \frac{\Delta\tau(y)}{120}$$

$$= \frac{3\tau^2(y)}{864} - \frac{\|\rho(y)\|^2}{720} + \frac{\|R(y)\|^2}{720} + \frac{\Delta\tau(y)}{120}$$

$$= \frac{\tau^2(y)}{288} - \frac{\|\rho(y)\|^2}{720} + \frac{\|R(y)\|^2}{720} + \frac{\Delta\tau(y)}{120}$$

i.e. $b_2(y,y) = \frac{1}{1440}(5\tau^2(y) - 2\,\|\rho(y)\|^2 + 2\,\|R(y)\|^2 + 12\Delta\tau(y)).$

<u>Remarks</u>

(1) The formula above is just the spectral quadratic polynomial:

$$\frac{1}{360}\,(2\,\|R\|^2 - 2\,\|\rho(R)\|^2 + 5\tau^2(R) + 12\Delta R)$$

given by A. Gray on page 342 of [21]'.

(2) M. Berger in [7] page 225 (E.IV.3) also gets:

$$\frac{1}{360}\left(5\tau^2(y) - 2\,\|\rho(y)\|^2 + 2\,\|R(y)\|^2\right) + \text{Constant } \Delta\tau(y).$$

The two formulae are just ours $\times\ \frac{1}{4}$. The extra factor of $\frac{1}{4}$ is due to the fact that we are dealing with $\frac{1}{2}\Delta$ and not Δ as do these authors.

(3) Notice that our formula is the same as:

$$\frac{1}{180}\left(10 \cdot \frac{1}{4}\left(\frac{\tau(y)}{2}\right)^2 - \frac{1}{4}\,\|\rho(y)\|^2 + \frac{1}{4}\,\|R(y)\|^2\right) + \frac{\Delta\tau(y)}{120}$$

which is the same as H.P. McKean and I.M. Singer's (see §7 of [30]) provided we take

$$\left(\frac{\tau(y)}{2}\right)^2 = 4A;\ \|\rho(y)\|^2 = 4B;\ \|R(y)\|^2 = 8C.$$

It is still easy to compute $b_2(y,y)$ when $b \not\equiv 0$ (at least when it is a gradient vector field). Thus we will assume that b is a gradient vector field and $V \equiv 0$.

Thus we go back to the "raw" expression for $b_2(y,y)$ given in (6.47) of §6:

$$b_2(y,y) = \frac{1}{2}\left(\frac{LC_y}{C_y}(y)\right)^2 + \frac{1}{12}\frac{\partial^2}{\partial x_\alpha^2}\frac{LC_y}{C_y}(y)$$

$$+ \frac{1}{4}\left\langle b(y) + \nabla C_y(y),\ \nabla\frac{LC_y}{C_y}(y)\right\rangle$$

$$= \frac{1}{2}\left(\frac{LC_y}{C_y}(y)\right)^2 + \frac{1}{12}\frac{\partial^2}{\partial x_\alpha^2}\frac{LC_y}{C_y}(y)$$

(since $\nabla C_y(y) = \nabla\theta_y^{-\frac{1}{2}}(y) + \nabla B_y(y) = -\,b(y)$)

$$= (I_1) + (I_2)$$

where

$$(I_1) = \frac{1}{2}\left(\frac{LC_y}{C_y}(y)\right)^2$$

and

$$(I_2) = \frac{1}{12}\,\frac{\partial^2}{\partial x_\alpha^2}\,\frac{LC_y}{C_y}(y)$$

$$(I_1) = \frac{1}{2}\left(\frac{I(y)}{12} - \frac{1}{2}\operatorname{div} b(y) - \frac{1}{2}\|b(y)\|^2\right)^2$$

by (6.26) of §6.

$$(I_2) = \frac{1}{12}\,(I_2') \quad \text{where}$$

$$(I_2') = \frac{\partial^2}{\partial x_\alpha^2}\,\frac{LC_y}{C_y}(y)$$

and $$\frac{LC_y}{C_y} = \frac{L(\theta_y^{-\frac12}\,B_y)}{\theta_y^{-\frac12}\,B_y}$$

$$= \frac{LB_y}{B_y} + \frac{L\theta_y^{-\frac12}}{\theta_y^{-\frac12}} + \langle \nabla \log B_y, \nabla \log \theta_y^{-\frac12}\rangle$$

$$= \frac{\frac12 \Delta B_y}{B_y} + \frac{\frac12 \Delta \theta_y^{-\frac12}}{\theta_y^{-\frac12}} + \langle b, \nabla \log B_y\rangle + \langle b, \nabla \log \theta_y^{-\frac12}\rangle$$

$$+ \langle \nabla \log B_y, \nabla \log \theta_y^{-\frac12}\rangle$$

$$= \frac12 B_y^{-1}\,\Delta B_y + \frac12\,\theta_y^{\frac12}\,\Delta \theta_y^{-\frac12} + \langle b, \nabla \log B_y\rangle$$

$$- \frac12 \langle b, \nabla \log \theta_y\rangle - \frac12 \langle \nabla \log B_y, \nabla \log \theta_y\rangle$$

$$= \frac12 B_y^{-1}\,\Delta B_y + \frac12\,\theta_y^{\frac12}\,\Delta \theta_y^{-\frac12} - \|b\|^2 = \frac{LC_y}{C_y}$$

(since b being a gradient vector field, we have $\nabla \log B_y = - b$).

Thus we have:

$$(I_2') = \frac{\partial^2}{\partial x_\alpha^2} \frac{LC_y}{C_y}(y) = \Delta(\frac{LC_y}{C_y})(y)$$

$$= \frac{1}{2}\,\Delta(B_y^{-1}\ \Delta B_y)(y) + \frac{1}{2}\,\Delta(\theta_y^{\frac{1}{2}}\Delta\theta_y^{-\frac{1}{2}})(y) - \Delta\,\|b\|^2\,(y)$$

$$= (I_{21}) + (I_{22}) + (I_{23})$$

$$(I_{21}) = \frac{1}{2}\,\Delta(B_y^{-1}\ \Delta By)(y)$$

$$= \frac{1}{2}\,\Delta\,B_y^{-1}(y)\cdot\Delta B_y(y) + \frac{1}{2}\,B_y^{-1}(y)\ \Delta\Delta B_y(y)$$

$$+ \langle\nabla\ B_y^{-1}(y),\ \nabla\Delta B_y(y)\rangle.$$

It is clear that

$$0 = \Delta(B_y^{-1}B_y)$$

$$= \Delta B_y^{-1}(y).B_y(y) + B_y^{-1}(y)\ \Delta B_y(y)$$

$$+ 2\langle\nabla\ B_y^{-1}(y),\ \nabla B_y(y)\rangle$$

$$= \Delta B_y^{-1}(y) + \Delta B_y(y) - 2\,\|b(y)\|^2.$$

Hence,

$$\Delta B_y^{-1}(y) = 2\,\|b(y)\|^2 - \Delta B_y(y)$$

and

$$(I_{21}) = (\,\|b(y)\|^2 - \tfrac{1}{2}\Delta B_y(y))\Delta B_y(y) + \tfrac{1}{2}\Delta\Delta B_y(y)$$

$$+ \langle b(y),\nabla\,\Delta\,B_y(y)\rangle.$$

Now,

$$\Delta B_y = - \operatorname{div} b + \|b\|^2$$

by (6.11) since b is a gradient vector field. Thus we have:

$$(I_{21}) = (\tfrac{1}{2}\|b(y)\|^2 + \tfrac{1}{2}\operatorname{div} b(y))(\|b(y)\|^2 - \operatorname{div} b(y))$$

$$+ \tfrac{1}{2}\Delta(- \operatorname{div} b + \|b\|^2)(y)$$

$$+ \langle b(y), \nabla(- \operatorname{div} b + \|b\|^2)(y)\rangle$$

$$= \tfrac{1}{2}(\|b(y)\|^4 - (\operatorname{div} b(y))^2) - \tfrac{1}{2}\Delta \operatorname{div} b(y) + \tfrac{1}{2}\Delta\|b\|^2(y)$$

$$- \langle b(y), \nabla \operatorname{div} b(y)\rangle + \langle b(y), \nabla\|b\|^2(y)\rangle.$$

Thus,

$$(I_2) = \tfrac{1}{12}(I_2') = \tfrac{1}{12}((I_{21}) + (I_{22}) + (I_{23}))$$

$$= \tfrac{1}{24}(\|b(y)\|^4 - (\operatorname{div} b(y))^2) - \tfrac{1}{24}\Delta \operatorname{div} b(y) + \tfrac{1}{24}\Delta\|b\|^2(y)$$

$$+ \tfrac{1}{12}\langle b(y), \nabla\|b\|^2(y) - \nabla \operatorname{div} b(y)\rangle$$

$$+ \tfrac{1}{24}\Delta(\theta_y^{\frac{1}{2}}\Delta\theta_y^{-\frac{1}{2}})(y) - \tfrac{1}{12}\Delta\|b\|^2(y).$$

Now,

$$\tfrac{1}{24}\Delta(\theta_y^{\frac{1}{2}}\Delta\theta_y^{-\frac{1}{2}})(y) = \tfrac{1}{24}\frac{\partial^2}{\partial x_\alpha^2}(\theta_y^{\frac{1}{2}}\Delta\theta_y^{-\frac{1}{2}})(y).$$

By (6.52) of §6,

$$b_2(y,y) = \frac{\tau^2(y)}{288} + \frac{1}{24}\frac{\partial^2}{\partial x_\alpha^2}(\theta_y^{\frac{1}{2}}\Delta\theta_y^{-\frac{1}{2}})(y)$$

$$= \frac{\tau^2(y)}{288} - \frac{\|\rho(y)\|^2}{720} + \frac{\|R(y)\|^2}{720} + \frac{\Delta\tau(y)}{120}$$

by the final computation of $b_2(y,y)$.

Thus we have:

$$\frac{1}{24}\frac{\partial^2}{\partial x_\alpha^2}(\theta_y^{\frac{1}{2}}\Delta\theta_y^{-\frac{1}{2}})(y) = -\frac{\|\rho(y)\|^2}{720} + \frac{\|R(y)\|^2}{720} + \frac{\Delta\tau(y)}{120}.$$

Consequently,

$$(I_2) = \frac{1}{24}(\|b(y)\|^4 - (\mathrm{div}\, b(y))^2) - \frac{1}{24}\Delta\,\mathrm{div}\, b(y)$$

$$- \frac{1}{24}\Delta\|b\|^2(y) + \frac{1}{12}\langle b(y), \nabla\|b\|^2(y) - \nabla\,\mathrm{div}\, b(y)\rangle$$

$$- \frac{\|\rho(y)\|^2}{720} + \frac{\|R(y)\|^2}{720} + \frac{\Delta\tau(y)}{120}.$$

Finally,

$$b_2(y,y) = (I_1) + (I_2)$$

$$= \frac{1}{2}(\frac{\tau(y)}{12} - \frac{1}{2}\mathrm{div}\, b(y) - \frac{1}{2}\|b(y)\|^2))^2 - \frac{1}{24}(\|b(y)\|^4 - (\mathrm{div}\, b(y))^2)$$

$$- \frac{1}{24}\Delta\,\mathrm{div}\, b(y) - \frac{1}{24}\Delta\|b\|^2(y) + \frac{1}{12}\langle b(y), \nabla\|b\|^2(y) - \nabla\mathrm{div}\, b(y)\rangle$$

$$- \frac{\|\rho(y)\|^2}{720} + \frac{\|R(y)\|^2}{720} + \frac{\Delta\tau(y)}{120}$$

§7. THE "RAW" EXPRESSION FOR THE FOURTH COEFFICIENT $b_3(y,y)$.

We end this (last) Chapter by giving what we call the "raw" expression for $b_3(y,y)$ i.e. it is an expression in terms of the derivatives of C_y and not in terms of the curvature of M at y as was the case of $b_1(y,y)$ and $b_2(y,y)$ in §6.

To shorten computations we shall assume that $b \equiv 0$ and $V \equiv 0$. We set $\phi = \theta_y^{-\frac{1}{2}}$. Thus LC_y becomes $L\phi$ and L_{C_y} is replaced by L_ϕ. We still write $L\phi$ instead of $\frac{1}{2}\Delta\phi$.

Recall that:

$$b_3(y,y) = \int_0^1 \int_0^{r_1} \int_0^{r_2} g_{r_1,r_2,r_3}(y)dr_1dr_2dr_3$$

where

$$g_{r_1,r_2,r_3}(y) = F(1,1-r_3)L_{C_y}F(1-r_3,1-r_2)L_{C_y}F(1-r_2,1-r_1)\frac{LC_y}{C_y}(y)$$

$$= L_{C_y}F(1-r_3,1-r_2)L_{C_y}F(1-r_2,1-r_1)\frac{LC_y}{C_y}(y)$$

$$= L_\phi F(1-r_3,1-r_2)L_\phi F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y)$$

$$= L(\phi F(1-r_1,1-r_2)L_\phi F(1-r_2,1-r_1)\frac{L\phi}{\phi})(y)$$

$$= L(F(1-r_3,1-r_2)L_\phi F(1-r_2,1-r_1)\frac{L\phi}{\phi})(y) \qquad (I_1)$$

$$+ F(1-r_3,1-r_2)L_\phi F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y)(\frac{L\phi}{\phi})(y) \qquad (I_2)$$

$$+ \langle\nabla\phi(y)) \nabla(F(1-r_3,1-r_2)L_\phi F(1-r_2,1-r_1)\frac{L\phi}{\phi})(y)\rangle. \qquad (I_3)$$

Now,

$$(I_1) = L(F(1-r_3,1-r_2)L_\phi F(1-r_2,1-r_1)\frac{L\phi}{\phi})(y).$$

Consider

$$L_\phi F(1-r_2,1-r_1)\frac{L\phi}{\phi} = \frac{L(\phi F(1-r_2,1-r_2)\frac{L\phi}{\phi})}{\phi}$$

$$= (F(1-r_2,1-r_1)\frac{L\phi}{\phi})(\frac{L\phi}{\phi}) + L(F(1-r_2,1-r_1)\frac{L\phi}{\phi})$$

$$+ \langle\nabla \log \phi, \nabla(F(1-r_2,1-r_1)\frac{L\phi}{\phi})\rangle.$$

Therefore,

$$(I_1) = L(F(1-r_3,1-r_2)((F(1-r_2,1-r_1)\frac{L\phi}{\phi})(\frac{L\phi}{\phi}))(y) \qquad (I_{11})$$

$$+ L(F(1-r_3,1-r_2)L(F(1-r_2,1-r_1)\frac{L\phi}{\phi}))(y) \qquad (I_{12})$$

$$+ L(F(1-r_3,1-r_2) \langle\nabla \log \phi, \nabla(F(1-r_2,1-r_1)\frac{L\phi}{\phi})\rangle)(y). \qquad (I_{13})$$

We next consider,

$$(I_{11}) = L(F(1-r_3,1-r_2)((F(1-r_2,1-r_1)\frac{L\phi}{\phi})(\frac{L\phi}{\phi})))(y)$$

$$= L(F(1-r_3,1-r_2)\cdot F(1-r_2,1-r_1)\frac{L\phi}{\phi})(y)(F(1-r_3,1-r_2)\frac{L\phi}{\phi})(y) \qquad (I_{111})$$

$$+ F(1-r_3,1-r_2)F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y)\cdot L(F(1-r_3,1-r_2)\frac{L\phi}{\phi})(y) \qquad (I_{112})$$

$$+ \langle\nabla F(1-r_3,1-r_2)F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y, \nabla(F(1-r_3,1-r_2)\frac{L\phi}{\phi})(y)\rangle. \qquad (I_{113})$$

Now,

Now,

$$(I_{111}) = L(F(1-r_3,1-r_2)F(1-r_2,1-r_1)\frac{L\phi}{\phi})(\frac{L\phi}{\phi})(y)$$

$$= (I'_{111})(\frac{L\phi}{\phi})(y)$$

where

$$(I'_{111}) = L(F(1-r_3,1-r_2)F(1-r_2,1-r_1)\frac{L\phi}{\phi})(y)$$

$$= \frac{1}{2}\,\Delta\;(F(1-r_3,1-r_2)F(1-r_2,1-r_1)\frac{L\phi}{\phi})(y)$$

$$+ \langle b(y),\nabla\; F(1-r_3,1-r_2)F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y)\rangle$$

$$= \frac{1}{2}\,g^{\alpha\beta}(y)\;[\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\,F(1-r_3,1-r_2)F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y)$$

$$- \Gamma^{\gamma}_{\alpha\beta}(y)\,\frac{\partial}{\partial x_\gamma}\,F(1-r_3,1-r_2)F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y)]$$

$$+ b^{\lambda}(y)\;g^{\gamma\lambda}(y)\,\frac{\partial}{\partial x_\lambda}\,F(1-r_3,1-r_2)F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y)$$

$$= \frac{1}{2}\,\frac{\partial^2}{\partial x_\alpha^2}\,F(1-r_3,1-r_2)F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y)$$

$$+ b^{\gamma}(y)\,\frac{\partial}{\partial x_\gamma}\,F(1-r_3,1-r_2)F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y)$$

(since $g^{\alpha\beta}(y) = \delta^{\alpha\beta}$ and $\Gamma^{\gamma}_{\alpha\beta}(y) = 0$).

$$= \frac{1}{2}\,\frac{\partial^2}{\partial x_\alpha^2}\,\frac{L\phi}{\phi}(y)\;\left(\frac{1-r_1}{1-r_2}\right)^2\left(\frac{1-r_2}{1-r_3}\right)^2$$

$$+ b^{\gamma}(y)\,\frac{\partial}{\partial x_\gamma}\;\frac{L\phi}{\phi}(y)\;\left(\frac{1-r_1}{1-r_2}\right)\left(\frac{1-r_2}{1-r_3}\right)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}\frac{L\phi}{\phi}(y)\left(\frac{1-r_1}{1-r_3}\right)^2 + b^\gamma(y)\frac{\partial}{\partial x_\gamma}\frac{L\phi}{\phi}(y)\left(\frac{1-r_1}{1-r_3}\right).$$

Thus,

$$(I_{111}) = \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}\frac{L\phi}{\phi}(y)\left(\frac{L\phi}{\phi}\right)(y)\left(\frac{1-r_1}{1-r_3}\right)^2$$

$$+ b^\gamma(y)\frac{\partial}{\partial x_\gamma}\frac{L\phi}{\phi}(y)\left(\frac{L\phi}{\phi}\right)(y)\left(\frac{1-r_1}{1-r_3}\right).$$

Next,

$$(I_{112}) = \frac{L\phi}{\phi}(y)\, L(F(1-r_3,1-r_2)\frac{L\phi}{\phi})(y)$$

$$= (I'_{112})\left(\frac{L\phi}{\phi}\right)(y)$$

where

$$(I'_{112}) = L(F(1-r_3,1-r_2)\frac{L\phi}{\phi})(y)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}F(1-r_3,1-r_2)\frac{L\phi}{\phi}(y) + b^\gamma(y)\frac{\partial}{\partial x_\gamma}F(1-r_3,1-r_2)\frac{L\phi}{\phi}(y)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}\frac{L\phi}{\phi}(y)\left(\frac{1-r_2}{1-r_3}\right)^2 + b^\gamma(y)\frac{\partial}{\partial x_\gamma}\frac{L\phi}{\phi}(y)\left(\frac{1-r_2}{1-r_3}\right).$$

Hence

$$(I_{112}) = \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}\frac{L\phi}{\phi}(y)\left(\frac{L\phi}{\phi}\right)(y)\left(\frac{1-r_2}{1-r_3}\right)^2$$

$$+ b^\gamma(y)\frac{\partial}{\partial x_\gamma}\frac{L\phi}{\phi}(y)\left(\frac{L\phi}{\phi}\right)(y)\left(\frac{1-r_2}{1-r_3}\right).$$

Consider:

$$(I_{113}) = \langle\nabla F(1-r_3,1-r_2)F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y),\ \nabla F(1-r_3,1-r_1)\frac{L\phi}{\phi}(y)\rangle$$

$$= \langle\nabla\frac{L\phi}{\phi}(y)\left(\frac{1-r_2}{1-r_3}\right)\left(\frac{1-r_1}{1-r_2}\right), \nabla\frac{L\phi}{\phi}(y)\left(\frac{1-r_2}{1-r_3}\right)\rangle$$

$$= \langle \nabla \frac{L\phi}{\phi}(y),\ \nabla \frac{L\phi}{\phi}(y)\rangle (\frac{1-r_1}{1-r_3})(\frac{1-r_2}{1-r_3})$$

$$= \|\nabla \frac{L\phi}{\phi}(y)\|^2 (\frac{(1-r_1)(1-r_2)}{(1-r_3)^2})$$

$$= \|\nabla \frac{L\phi}{\phi}(y)\|^2 \ \frac{(1-r_1)(1-r_2)}{(1-r_3)^2}\ .$$

Next consider:

$$(I_{12}) = L(F(1-r_3,1-r_2)\cdot L(F(1-r_2,1-r_1)\frac{L\phi}{\phi})(y)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2} L(F(1-r_2,1-r_1)\frac{L\phi}{\phi})(y)(\frac{1-r_2}{1-r_3})^2$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}[\frac{1}{2} g^{\beta\gamma}(\frac{\partial^2}{\partial x_\beta \partial x_\gamma} F(1-r_2,1-r_1)\frac{L\phi}{\phi} - \Gamma^\lambda_{\beta\gamma}\frac{\partial}{\partial x_\lambda}F(1-r_2,1-r_1)\frac{L\phi}{\phi}](y)(\frac{1-r_2}{1-r_3})^2$$

$$= \frac{1}{4}\frac{\partial^2}{\partial x_\alpha^2}[g^{\beta\gamma}\ \frac{\partial^2}{\partial x_\beta \partial x_\gamma} F(1-r_2,1-r_1)\frac{L\phi}{\phi}](y)(\frac{1-r_2}{1-r_3})^2 \qquad (I_{121})$$

$$- \frac{1}{4}\frac{\partial^2}{\partial x_\alpha^2}[g^{\beta\gamma}\ \Gamma^\lambda_{\beta\gamma}\ \frac{\partial}{\partial x_\lambda} F(1-r_2,1-r_1)\frac{L\phi}{\phi}](y)(\frac{1-r_2}{1-r_2})^2 \qquad (I_{122})$$

where

$$(I_{121}) = \frac{1}{4}\frac{\partial^2}{\partial x_\alpha^2}[g^{\beta\gamma}\ \frac{\partial^2}{\partial x_\beta \partial x_\gamma} F(1-r_2,1-r_1)\frac{L\phi}{\phi}](y)(\frac{1-r_2}{1-r_3})^2$$

$$= \frac{1}{4}\frac{\partial^2}{\partial x^2}[g^{\beta\gamma}\ F(1-r_2,1-r_1)\ \frac{\partial^2}{\partial x_\beta \partial x_\gamma}\ \frac{L\phi}{\phi}](y)(\frac{1-r_2}{1-r_3})^2(\frac{1-r_1}{1-r_2})^2$$

$$= \frac{1}{4}[\frac{\partial^2 g}{\partial x_\alpha^2}(y)F(1-r_2,1-r_1)\frac{\partial^2}{\partial x_\beta \partial x_\gamma}\ \frac{L\phi}{\phi}(y)](\frac{1-r_1}{1-r_3})^2$$

$$+ \frac{1}{4}[g^{\beta\gamma}(y)\ \frac{\partial^2}{\partial x_\alpha^2}(F(1-r_2,1-r_1)\frac{\partial^2}{\partial x_\beta \partial x_\gamma}\ \frac{L\phi}{\phi})](y)(\frac{1-r_1}{1-r_3})^2$$

$$+ \frac{1}{2}[\frac{\partial g^{\beta\gamma}}{\partial x_\alpha}(y)\ \frac{\partial}{\partial x_\alpha}(F(1-r_2,1-r_1)\frac{\partial^2}{\partial x_\beta \partial x_\gamma}\ \frac{L\phi}{\phi})(y)](\frac{1-r_1}{1-r_3})^2$$

$$= \frac{1}{4}\left[\frac{\partial^2 q}{\partial x_\alpha^2}(y)\ \frac{\partial^2}{\partial x_\beta \partial x_\gamma}\ \frac{L\phi}{\phi}(y)\right]\left(\frac{1-r_1}{1-r_3}\right)^2 \qquad (I_{1211})$$

$$+ \frac{1}{4}\left[\frac{\partial^4}{\partial x_\alpha^2\ \partial x_\beta^2}\ \frac{L\phi}{\phi}\right](y)\left(\frac{1-r_1}{1-r_3}\right)^2\left(\frac{1-r_1}{1-r_2}\right)^2 \qquad (I_{1212})$$

since

$$g^{\beta\gamma}(y) = \delta^{\beta\gamma} \quad \text{and} \quad \frac{\partial g^{\beta\gamma}}{\partial x_\alpha}(y) = 0$$

$$= (I_{1211}) + (I_{1212})$$

$$(I_{122}) = -\frac{1}{4}\frac{\partial^2}{\partial x_\alpha^2}\left[g^{\beta\gamma}\ \Gamma^\lambda_{\beta\gamma}\ \frac{\partial}{\partial x_\lambda}\left(F(1-r_2,1-r_1)\frac{L\phi}{\phi}\right)\right](y)\left(\frac{1-r_2}{1-r_3}\right)^2$$

$$= -\frac{1}{4}\frac{\partial^2}{\partial x_\alpha^2}\left[g^{\beta\gamma}\ \Gamma^\lambda_{\beta\gamma} F(1-r_2,1-r_1)\ \frac{\partial}{\partial x_\lambda}\ \frac{L\phi}{\phi}\right](y)\left(\frac{1-r_2}{1-r_3}\right)^2\left(\frac{1-r_1}{1-r_2}\right)$$

$$= -\frac{1}{4}\left[\frac{\partial^2}{\partial x_\alpha^2}\ (g^{\beta\gamma}\ \Gamma^\lambda_{\beta\gamma})(y).F(1-r_2,1-r_1)\frac{\partial}{\partial x_\lambda}\ \frac{L\phi}{\phi}(y)\right.$$

$$+ g^{\beta\gamma}(y)\Gamma^\lambda_{\beta\gamma}(y)\frac{\partial^2}{\partial x_\alpha^2}\ \left(F(1-r_2,1-r_1)\frac{\partial}{\partial x_\lambda}\ \frac{L\phi}{\phi}\right)(y)$$

$$\left. + 2\ \frac{\partial}{\partial x_\alpha}(g^{\beta\gamma}\ \Gamma^\lambda_{\beta\gamma})(y)\ \frac{\partial}{\partial x_\alpha}\ \left(F(1-r_2,1-r_1)\frac{\partial}{\partial x_\lambda}\ \frac{L\phi}{\phi}\right)(y)\right]\frac{(1-r_1)(1-r_2)}{(1-r_3)^2}$$

$$= -\frac{1}{4}\left[\frac{\partial^2}{\partial x_\alpha^2}\ (g^{\beta\gamma}\ \Gamma^\lambda_{\beta\gamma})(y)\ \frac{\partial}{\partial x_\lambda}\ \frac{L\phi}{\phi}(y)\right]\frac{(1-r_1)(1-r_2)}{(1-r_3)^2} \qquad (I_{1221})$$

$$-\frac{1}{2}\left[\frac{\partial \Gamma^\lambda_{\beta\beta}}{\partial x_\alpha}(y)\ \frac{\partial^2}{\partial x_\alpha \partial x_\lambda}\ \frac{L\phi}{\phi}(y)\right]\left(\frac{1-r_1}{1-r_3}\right)^2 \qquad (I_{1222})$$

$$= (I_{1221}) + (I_{1222}).$$

We next compute (I_{13}):

$$(I_{13}) = L(F(1-r_3,1-r_2)\langle\nabla \log \phi,\ \nabla(F(1-r_2,1-r_1)\frac{L\phi}{\phi})\rangle)(y)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2} F(1-r_3,1-r_2)\langle\nabla \log \phi,\ \nabla(F(1-r_2,1-r_1)\frac{L\phi}{\phi})\rangle(y)$$

$$+ b^\gamma(y)\frac{\partial}{\partial x_\gamma} F(1-r_3,1-r_2)\langle\nabla \log \phi,\ \nabla(F(1-r_2,1-r_1)\frac{L\phi}{\phi})\rangle(y)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}\langle\nabla \log \phi,\ \nabla(F(1-r_2,1-r_1)\frac{L\phi}{\phi})\rangle(y)\left(\frac{1-r_2}{1-r_3}\right)^2 \qquad (I_{131})$$

$$+ b^\gamma(y)\frac{\partial}{\partial x_\gamma}\langle\nabla \log \phi,\ \nabla(F(1-r_2,1-r_1)\frac{L\phi}{\phi})\rangle(y)\left(\frac{1-r_2}{1-r_3}\right) \qquad (I_{132})$$

$$(I'_{131}) = \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}\langle\nabla \log \phi,\ \nabla(F(1-r_2,1-r_1)\frac{L\phi}{\phi})\rangle(y)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}\langle g^{\delta\beta}\frac{\partial}{\partial x_\beta}(\log \phi)\frac{\partial}{\partial x_\delta},\ g^{\gamma\lambda}\frac{\partial}{\partial x_\lambda}(F(1-r_2,1-r_1)\frac{L\phi}{\phi})\frac{\partial}{\partial x_\gamma}\rangle(y)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}[g^{\delta\beta}\frac{\partial}{\partial x_\beta}(\log \phi)\ g^{\gamma\lambda}\frac{\partial}{\partial x_\lambda}(F(1-r_2,1-r_1)\frac{L\phi}{\phi})\langle\frac{\partial}{\partial x_\delta},\frac{\partial}{\partial x_\gamma}\rangle](y)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}[g^{\delta\beta}\ g^{\gamma\lambda}\ g_{\delta\gamma}\frac{\partial}{\partial x_\beta}(\log \phi)\frac{\partial}{\partial x_\lambda}(F(1-r_2,1-r_1)\frac{L\phi}{\phi})](y)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}[\delta^\beta_\gamma\ g^{\gamma\lambda}\frac{\partial}{\partial x_\beta}(\log \phi)\frac{\partial}{\partial x_\lambda}(F(1-r_2,1-r_1)\frac{L\phi}{\phi})](y)$$

(since $g^{\delta\beta}\,g_{\delta\gamma} = \delta^\beta_\gamma$)

$$= \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}[g^{\beta\lambda}\frac{\partial}{\partial x_\beta}(\log \phi)\frac{\partial}{\partial x_\lambda}(F(1-r_2,1-r_1)\frac{L\phi}{\phi})](y)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}[g^{\beta\lambda}\frac{\partial}{\partial x_\beta}(\log \phi)\ F(1-r_2,1-r_1)\frac{\partial}{\partial x_\lambda}\frac{L\phi}{\phi}](y)\left(\frac{1-r_1}{1-r_2}\right)$$

$$= \frac{1}{2}[\frac{\partial^2}{\partial x_\alpha^2}(g^{\beta\lambda}\frac{\partial}{\partial x_\beta}(\log \phi))F(1-r_2,1-r_1)\frac{\partial}{\partial x_\lambda}\frac{L\phi}{\phi}$$

$$+ g^{\beta\lambda} \frac{\partial}{\partial x_\beta} (\log \phi) \frac{\partial^2}{\partial x_\alpha^2} (F(1-r_2, 1-r_1) \frac{\partial}{\partial x_\lambda} \frac{L\phi}{\phi})$$

$$+ 2 \frac{\partial}{\partial x_\alpha} (g^{\beta\lambda} \frac{\partial}{\partial x_\beta} (\log \phi)) \frac{\partial}{\partial x_\alpha} (F(1-r_2, 1-r_1) \frac{\partial}{\partial x_\lambda} \frac{L\phi}{\phi}](y)(\frac{1-r_1}{1-r_2})$$

$$= \frac{1}{2} [\frac{\partial^2}{\partial x_\alpha^2} (g^{\beta\lambda} \frac{\partial}{\partial x_\beta} (\log \phi))(y) \frac{\partial}{\partial x_\lambda} \frac{L\phi}{\phi}(y)$$

$$+ 2 \frac{\partial}{\partial x_\alpha} (g^{\beta\lambda} \frac{\partial}{\partial x_\beta} (\log \phi))(y) \frac{\partial^2}{\partial x_\alpha \partial x_\lambda} \frac{L\phi}{\phi}(y)(\frac{1-r_1}{1-r_2})](\frac{1-r_1}{1-r_2})$$

(since $\frac{\partial}{\partial x_\beta} (\log \phi)(y) = 0$)

$$= (I_{1311}) + (I_{1312}) \text{ where}$$

$$(I_{1311}) = \frac{1}{2} [\frac{\partial^2}{\partial x_\alpha^2} (g^{\beta\lambda} \frac{\partial}{\partial x_\beta} (\log \phi))(y) \frac{\partial}{\partial x_\lambda} \frac{L\phi}{\phi}(y)](\frac{1-r_1}{1-r_2})$$

$$= \frac{1}{2} [\frac{\partial^2 g^{\beta\lambda}}{\partial x_\alpha^2}(y) \cdot \frac{\partial}{\partial x_\beta} (\log \phi)(y) + g^{\beta\lambda}(y) \frac{\partial^3}{\partial x_\alpha^2 \partial x_\beta} (\log \phi)(y)$$

$$+ 2 \frac{\partial g^{\beta\lambda}}{\partial x_\alpha}(y) \frac{\partial^2}{\partial x_\alpha \partial x_\beta} (\log \phi)(y)] \frac{\partial}{\partial x_\lambda} \frac{L\phi}{\phi}(y)(\frac{1-r_1}{1-r_2})$$

$$= \frac{1}{2} [\frac{\partial^3}{\partial x_\alpha^2 \partial x_\lambda} (\log \phi)(y)] [\frac{\partial}{\partial x_\lambda} \frac{L\phi}{\phi}(y)](y)(\frac{1-r_1}{1-r_2})$$

$$(I_{1312}) = [\frac{\partial}{\partial x_\alpha} (g^{\beta\lambda} \frac{\partial}{\partial x_\beta} (\log \phi))(y)][\frac{\partial^2}{\partial x_\alpha \partial x_\lambda} \frac{L\phi}{\phi}(y)](\frac{1-r_1}{1-r_2})^2$$

$$= [\frac{\partial g^{\beta\lambda}}{\partial x_\alpha}(y) \frac{\partial}{\partial x_\beta} (\log \phi)(y) + g^{\beta\lambda}(y) \frac{\partial^2}{\partial x_\alpha \partial x_\beta} (\log \phi)(y)] \times$$

$$[\frac{\partial^2}{\partial x_\alpha \partial x_\lambda} \frac{L\phi}{\phi}(y)](\frac{1-r_1}{1-r_2})^2$$

$$= [\frac{\partial^2}{\partial x_\alpha \partial x_\lambda} (\log \phi)(y)][\frac{\partial^2}{\partial x_\alpha \partial x_\lambda} \frac{L\phi}{\phi}(y)](\frac{1-r_1}{1-r_2})^2.$$

Thus we get:

$$(I'_{131}) = (I_{1311}) + (I_{1312})$$

and so:

$$I_{131} = (I'_{131})(\frac{1-r_2}{1-r_3})^2$$

$$= \frac{1}{2}[-\frac{\partial^3}{\partial x_\alpha^2 \partial x_\lambda}(\log \phi)(y) \frac{\partial}{\partial x_\lambda} \frac{L\phi}{\phi}(y)] \frac{(1-r_1)(1-r_2)}{(1-r_3)^2}$$

$$+ \frac{\partial^2}{\partial x_\alpha \partial x_\lambda}(\log \phi)(y) \frac{\partial^2}{\partial x_\alpha \partial x_\lambda} \frac{L\phi}{\phi}(y)(\frac{1-r_1}{1-r_3})^2$$

Next consider:

$$(I_{132}) = (I'_{132})b^\gamma(y)(\frac{1-r_2}{1-r_3})$$

where

$$(I'_{132}) = \frac{\partial}{\partial x_\gamma} \langle \nabla \log \phi, \nabla(F(r_2,1-r_1)\frac{L\phi}{\phi})\rangle(y)$$

$$= \frac{\partial}{\partial x_\gamma}[g^{\beta\lambda} \frac{\partial}{\partial x_\beta} (\log \phi) F(1-r_2,1-r_1)\frac{\partial}{\partial x_\lambda} \frac{L\phi}{\phi}](y)(\frac{1-r_1}{1-r_2})$$

$$= [\frac{\partial}{\partial x_\gamma}(g^{\beta\lambda} \frac{\partial}{\partial x_\beta}(\log \phi))(y) F(1-r_2,1-r_1) \frac{\partial}{\partial x_\lambda} \frac{L\phi}{\phi}(y)$$

$$+ g^{\beta\lambda}(y) \frac{\partial}{\partial x_\beta}(\log \phi)(y) \frac{\partial}{\partial x_\gamma}(F(1-r_2,1-r_1)\frac{\partial}{\partial x_\gamma} \frac{L\phi}{\phi})(y)](\frac{1-r_1}{1-r_2})$$

$$= [(\frac{\partial g^{\beta\lambda}}{\partial x_\gamma}(y) \frac{\partial}{\partial x_\beta}(\log \phi)(y) + g^{\beta\lambda}(y)\frac{\partial^2}{\partial x_\gamma \partial x_\beta}(\log \phi)(y)) \frac{\partial}{\partial x_\lambda} \frac{L\phi}{\phi}(y)](\frac{1-r_1}{1-r_2})$$

(since $\frac{\partial}{\partial x_\beta}(\log \phi)(y) = 0$)

$$= [\frac{\partial^2}{\partial x_\gamma \partial x}(\log \phi)(y) \frac{\partial}{\partial x_\lambda} \frac{L\phi}{\phi}(y)](\frac{1-r_1}{1-r_2})$$

(since $\frac{\partial g^{\beta\lambda}}{\partial x_\gamma}(y) = 0$)

and so we get:

$$(I_{132}) = b^{\gamma}(y)\left(\frac{\partial^2}{\partial x_\gamma \partial x_\lambda}(\log \phi)(y)\right)\left(\frac{\partial}{\partial x_\lambda}\frac{L\phi}{\phi}(y)\right)\left(\frac{1-r_1}{1-r_3}\right).$$

Thus we obtain:

$$(I_{13}) = (I_{131}) + (I_{132})$$

$$= (I_{1311}) + (I_{1312}) + (I_{132})$$

$$= \frac{1}{2}\left[\frac{\partial^3}{\partial x_\alpha^2 \partial x_\lambda}(\log \phi)(y)\ \frac{\partial}{\partial x_\lambda}\frac{L\phi}{\phi}(y)\right]\frac{(1-r_1)(1-r_2)}{(1-r_3)^2}$$

$$+ \frac{\partial^2}{\partial x_\alpha \partial x_\lambda}(\log \phi)(y)\ \frac{\partial^2}{\partial x_\alpha \partial x_\lambda}\frac{L\phi}{\phi}(y)\left(\frac{1-r_1}{1-r_3}\right)^2$$

$$+ b^{\gamma}(y)\left(\frac{\partial^2}{\partial x_\gamma \partial x_\lambda}(\log \phi)(y)\left(\frac{\partial}{\partial x_\lambda}\frac{L\phi}{\phi}(y)\right)\left(\frac{1-r_1}{1-r_3}\right).$$

Thus we obtain:

$$(I_1) = (I_{11}) + (I_{12}) + (I_{13}).$$

We next compute (I_2):

$$(I_2) = F(1-r_3,1-r_2)L_\phi F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y)\left(\frac{L\phi}{\phi}\right)(y)$$

$$= L_\phi F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y)\left(\frac{L\phi}{\phi}\right)(y)$$

$$= (I_2')\left(\frac{L\phi}{\phi}\right)(y)$$

where $(I_2') = L_\phi F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y)$

$$= F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y)\left(\frac{L\phi}{\phi}\right)(y) + L(F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y)$$

$$+ \langle \nabla \log \phi(y),\ \nabla(F(1-r_2,1-r_1)\frac{L\phi}{\phi}(y)\rangle$$

$$= \left(\frac{L\phi}{\phi}(y)\right)^2 + L\left(F(1-r_2,1-r_1)\frac{L\phi}{\phi}\right)(y)$$

(since $\nabla \log \phi(y) = 0$)

$$= \left(\frac{L\phi}{\phi}(y)\right)^2 + \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}\left(F(1-r_2,1-r_1)\frac{L\phi}{\phi}\right)(y)$$

$$= \left(\frac{L\phi}{\phi}(y)\right)^2 + \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}\frac{L\phi}{\phi}(y)\left(\frac{1-r_1}{1-r_2}\right)^2.$$

Hence we have:

$$(I_2) = (I_2')\,\frac{L\phi}{\phi}(y)$$

$$= \left(\frac{L\phi}{\phi}(y)\right)^3 + \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}\frac{L\phi}{\phi}(y)\,\frac{L\phi}{\phi}(y)\left(\frac{1-r_1}{1-r_2}\right)^2.$$

Clearly $(I_3) = 0$

(since $\nabla\phi(y) = 0$)

and so

$$g_{r_1,r_2,r_3}(y) = (I_1) + (I_2) + (I_3)$$

$$= (I_1) + (I_2)$$

$$= (I_{11}) + (I_{12}) + (I_{13}) + (I_2)$$

$$= (I_{111}) + (I_{112}) + (I_{113}) + (I_{121}) + (I_{122}) + (I_{13}) + (I_2)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}\frac{L\phi}{\phi}(y)\,\frac{L\phi}{\phi}(y)\left(\frac{1-r_1}{1-r_3}\right)^2$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}\frac{L\phi}{\phi}(y)\,\frac{L\phi}{\phi}(y)\left(\frac{1-r_2}{1-r_3}\right)^2$$

$$+ \left\|\nabla\frac{L\phi}{\phi}(y)\right\|^2\frac{(1-r_1)(1-r_2)}{(1-r_3)^2}$$

$$+ \frac{1}{4}\left[\frac{\partial^2 g^{\beta\gamma}}{\partial x_\alpha^2}(y)\frac{\partial^2}{\partial x_\beta \partial x_\gamma}\frac{L\phi}{\phi}(y)\right]\left(\frac{1-r_1}{1-r_3}\right)^2$$

$$+ \frac{1}{4}\left[\frac{\partial^4}{\partial x_\alpha^2\, \partial x_\beta^2}\frac{L\phi}{\phi}(y)\right]\left(\frac{1-r_1}{1-r_3}\right)^2\left(\frac{1-r_1}{1-r_2}\right)^2$$

$$- \frac{1}{4}\left[\frac{\partial^2}{\partial x_\alpha^2}(g^{\beta\gamma}\ \Gamma^\lambda_{\beta\gamma})(y) \qquad \frac{\partial}{\partial x_\lambda}\frac{L\phi}{\phi}(y)\right]\frac{(1-r_1)(1-r_2)}{(1-r_3)^2}$$

$$- \frac{1}{2}\left[\frac{\partial \Gamma^\lambda_{\beta\beta}}{\partial x_\alpha}(y)\ \frac{\partial^2}{\partial x_\alpha \partial x_\lambda}\frac{L\phi}{\phi}(y)\right]\left(\frac{1-r_1}{1-r_3}\right)^2$$

$$+ \frac{1}{2}\left[\frac{\partial^3}{\partial x_\alpha^2 \partial x_\lambda}(\log\phi)(y)\ \frac{\partial}{\partial x_\lambda}\frac{L\phi}{\phi}(y)\right]\frac{(1-r_1)(1-r_2)}{(1-r_3)^2}$$

$$+ \frac{\partial^2}{\partial x_\alpha \partial x_\lambda}(\log\phi)(y)\ \frac{\partial^2}{\partial x_\alpha \partial x_\lambda}\frac{L\phi}{\phi}(y)\left(\frac{1-r_1}{1-r_3}\right)^2$$

$$+ \left(\frac{L\phi}{\phi}(y)\right)^3 + \frac{1}{2}\frac{\partial^2}{\partial x_\alpha^2}\frac{L\phi}{\phi}(y)\ \frac{L\phi}{\phi}(y)\left(\frac{1-r_1}{1-r_2}\right)^2.$$

Easy integrations show that:

$$\int_0^1\int_0^{r_1}\int_0^{r_2}\left(\frac{1-r_1}{1-r_3}\right)^3 dr_1dr_2dr_3 = \frac{1}{36}$$

$$\int_0^1\int_0^{r_1}\int_0^{r_2}\left(\frac{1-r_2}{1-r_3}\right)^3 dr_1dr_2dr_3 = \frac{1}{4}$$

$$\int_0^1\int_0^{r_1}\int_0^{r_2}\frac{(1-r_1)(1-r_2)}{(1-r_3)^2} dr_1dr_2dr_3 = \frac{1}{24}$$

$$\int_0^1\int_0^{r_1}\int_0^{r_2}\left(\frac{1-r_1}{1-r_2}\right)^2 dr_1dr_2dr_3 = \frac{1}{18}$$

$$\int_0^1\int_0^{r_1}\int_0^{r_2}\left(\frac{1-r_1}{1-r_2}\right)^2\left(\frac{1-r_1}{1-r_3}\right)^2 dr_1dr_2dr_3 = \frac{1}{60}\ .$$

Thus,

$$b_3(y,y) = \int_0^1 \int_0^{r_1} \int_0^{r_2} g_{r_1,r_2,r_3}(y)dr_1dr_2dr_3$$

$$= \frac{1}{72} \frac{\partial^2}{\partial x_\alpha^2} \frac{L\phi}{\phi}(y) \frac{L\phi}{\phi}(y)$$

$$+ \frac{1}{8} \frac{\partial^2}{\partial x_\alpha^2} \frac{L\phi}{\phi}(y) \frac{L\phi}{\phi}(y) + \frac{1}{24} \|\nabla \frac{L\phi}{\phi}(y)\|^2$$

$$+ \frac{1}{144} [\frac{\partial^2 g^{\beta\gamma}}{\partial x_\alpha^2}(y) \frac{\partial^2}{\partial x_\beta \partial x_\gamma} \frac{L\phi}{\phi}(y)]$$

$$+ \frac{1}{240}[\frac{\partial^4}{\partial x_\alpha^2 \partial x_\beta^2} \frac{L\phi}{\phi}(y)]$$

$$- \frac{1}{96}[\frac{\partial^2}{\partial x^2}(g^{\beta\gamma} \Gamma^\lambda_{\beta\gamma})(y) \frac{\partial}{\partial x_\lambda} \frac{L\phi}{\phi}(y)]$$

$$- \frac{1}{72}[\frac{\partial \Gamma^\lambda_{\beta\beta}}{\partial x_\alpha}(y) \frac{\partial^2}{\partial x_\alpha \partial x_\lambda} \frac{L\phi}{\phi}(y)]$$

$$+ \frac{1}{48}[\frac{\partial^2}{\partial x_\alpha^2 \partial x_\lambda}(\log \phi)(y) \frac{\partial}{\partial x_\lambda} \frac{L\phi}{\phi}(y)]$$

$$+ \frac{1}{36}[\frac{\partial^2}{\partial x_\alpha \partial x_\lambda}(\log \phi)(y) \frac{\partial^2}{\partial x_\alpha \partial x_\lambda} \frac{L\phi}{\phi}(y)]$$

$$+ \frac{1}{6}(\frac{L\phi}{\phi}(y))^3 + \frac{1}{36}[\frac{\partial^2}{\partial x_\alpha^2} \frac{L\phi}{\phi}(y) \frac{L\phi}{\phi}(y)]$$

$$= \frac{1}{6} \frac{\partial^2}{\partial x^2} \frac{L\phi}{\phi}(y) \frac{L\phi}{\phi}(y) + \frac{1}{24} (\frac{\partial}{\partial x_\alpha} \frac{L\phi}{\phi}(y))^2$$

$$+ \frac{1}{144}[\frac{\partial^2 g^{\beta\gamma}}{\partial x_\alpha^2}(y) \frac{\partial^2}{\partial x_\beta \partial x_\gamma} \frac{L\phi}{\phi}(y)] + \frac{1}{240} [\frac{\partial^4}{\partial x_\alpha^2 \partial x_\beta^2} \frac{L\phi}{\phi}(y)]$$

$$- \frac{1}{96} [\frac{\partial^2}{\partial x_\alpha^2}(g^{\beta\gamma} \Gamma^\lambda_{\beta\gamma})(y) \frac{\partial}{\partial x_\lambda} \frac{L\phi}{\phi}(y)] - \frac{1}{72} [\frac{\partial \Gamma^\lambda_{\beta\beta}}{\partial x_\alpha}(y) \frac{\partial^2}{\partial x_\alpha \partial x_\lambda} \frac{L\phi}{\phi}(y)]$$

$$+ \frac{1}{48} [\frac{\partial^3}{\partial x_\alpha^2 \partial x_\lambda}(\log \phi)(y) \frac{\partial}{\partial x_\lambda} \frac{L\phi}{\phi}(y)]$$

$$+ \frac{1}{36} [\frac{\partial^2}{\partial x_\alpha \partial x_\lambda}(\log \phi)(y) \frac{\partial^2}{\partial x_\alpha \partial x_\lambda} \frac{L\phi}{\phi}(y)] + \frac{1}{6} (\frac{L\phi}{\phi}(y))^3.$$

The above is the "raw" expression for $b_3(y,y)$ when $b \equiv 0$ and $V \equiv 0$. The geometric significance is a straightforward computation in terms of the curvature at y but it is too long to be included here.

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